Existence and Uniqueness Theorem

Just like for first order equations, we have an existence and uniqueness theorem for second-order linear equations.

Theorem. *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t) \qquad y(t_0) = y_0,\ y'(t_0) = y_0',$$

where p, q, and g are continuous functions on an open interval I containing t_0. This problem has exactly one solution $y = \phi(t)$, and this solution exists throughout the interval I.

- Should remind you of the theorem from first order linear equations.
- Can find interval of existence by finding the discontinuities of p, q, g and expanding out from t_0.

→ Problems are the same as for first-order.

Fundamental Set of Solutions

Now that we know solutions exist and are unique to these equations, we can talk about fundamental sets (or a basis) of solutions.

Definition. We say that y_1 and y_2 form a **fundamental set** of solutions to the differential equation $y'' + p(t)y' + q(t)y = 0$ if the set of all linear combinations $c_1y_1 + c_2y_2$ contains all solutions to the differential equation.

Note: y_1, y_2 must already be solutions [$c_1 = 1, c_2 = 0$]

• Means that the linear combination $c_1y_1 + c_2y_2$ can meet all initial conditions at a given point.

→ This happens when

$$\boxed{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0}$$

↳ Equivalent definition of Fundamental set.

Extra Terminology

- If y_1, y_2 are two solutions to a linear second-order, homogeneous equation, then we define

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

as the Wronskian of these two solutions.

- Two solutions y_1, y_2 are linearly independent if $W(t) \neq 0$.

- For any two solutions, either $W(t) = 0$ for all t. or $W(t) \neq 0$ for all t.