Euler's Formula and Complex Numbers

For the next case of roots of the characteristic equation for second-order, linear, constant-coefficient equations, we will look at complex roots. This needs some properties of complex numbers.

$$ar^2 + br + c = 0$$

If $b^2 - 4ac < 0 \rightarrow$ Two complex roots.

Basic Properties of Complex Numbers

$$z = x + iy \qquad i = \sqrt{-1}$$

$$\mathrm{Re}(z) = x \qquad \mathrm{Im}(z) = y$$

$$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$$

$$(a + ib)(c + id) = ac + ibc + iad + i^2 bd$$
$$= \underline{(ac - bd) + i(ad + bc)}$$

$$\frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

$$\overline{a+ib} = a - ib \qquad |a+ib| = \sqrt{a^2+b^2}$$

In order to solve constant-coefficient equations with complex roots, we need to determine what e^{rt} is, for r a complex number. This comes down to Euler's formula.

$r = a + ib$

$$e^{rt} = e^{(a+ib)t} = e^{at+ibt}$$

$$\boxed{= \underbrace{e^{at}} \cdot e^{ibt}}$$

Euler's Formula

$$e^{ix} = \cos(x) + i\sin(x)$$

→ Can be proved using Taylor Series

$$\boxed{e^{rt} = e^{at}\left(\cos(bt) + i\sin(bt)\right)}$$