Amplitude of the Steady-State Response

Now, we want to figure out how the amplitude of this steady state solution depends on the forcing amount, along with other factors. We can work in general to compute this amplitude.

$$ay'' + by' + cy = F_0 \cos(\omega t)$$

$$\boxed{R}\cos(\omega t - \varphi)$$

Guess $A\cos(\omega t) + B\sin(\omega t)$

$$a\left[-A\omega^2\cos(\omega t) - B\omega^2 \sin \omega t\right] +$$

$$b\left[-A\omega \sin(\omega t) + B\omega \cos(\omega t)\right] +$$

$$c\left[A\cos(\omega t) + B\sin(\omega t)\right] =$$

$$\cos(\omega t)\left[-aA\omega^2 + bB\omega + cA\right] +$$

$$\sin(\omega t)\left[-aB\omega^2 - bA\omega + cB\right] \overset{!}{=} F_0 \cos(\omega t)$$

$$\left[A(-a\omega^2+c) + B(b\omega) = F_0\right] b\omega$$

$$+\left(A(-b\omega) + B(-a\omega^2+c) = 0\right](c-a\omega^2)$$

$$B(b\omega)^2 + B(c-a\omega^2)^2 = F_0 b\omega$$

$$\boxed{B = \frac{F_0 b\omega}{(b\omega)^2 + (c-a\omega^2)^2}}$$

$$A(-b\omega) + \frac{F_0 b\omega}{(b\omega)^2+(c-a\omega^2)^2}\cdot(c-a\omega^2) = 0$$

$$\boxed{A = \frac{F_0(c-a\omega^2)}{(b\omega)^2 + (c-a\omega^2)^2}}$$

$$R = \sqrt{A^2+B^2}$$

$$R = \sqrt{\frac{F_0^2(c-a\omega^2)^2}{\left[(b\omega)^2+(c-a\omega^2)^2\right]^2} + \frac{F_0^2(b\omega)^2}{\left[(b\omega)^2+(c-a\omega^2)^2\right]^2}}$$

$$R = F_0 \sqrt{\frac{1}{(bw)^2 + (c - aw^2)^2}}$$

$$\frac{R}{F_0/c}$$

R ← Amplitude of the resulting osillation

F_0/c ← Length a force of F_0 will move the spring.

$$\frac{Rc}{F_0} = c \sqrt{\frac{1}{(bw)^2 + (c - aw^2)^2}}$$

$w_0^2 = \frac{c}{a}$ → $c - aw^2 = c\left(1 - \frac{a}{c} w^2\right)$

$$= c\left(1 - \frac{w^2}{w_0^2}\right)$$

$$= \sqrt{\frac{c^2}{b^2 w^2 + c^2\left(1 - \frac{w^2}{w_0^2}\right)^2}}$$

$$w_0^2 = \frac{c}{a} \qquad b^2 w^2 = \frac{c}{a} \cdot b^2 \cdot \frac{w^2}{w_0^2}$$

$$\frac{Rc}{F_0} = \sqrt{\frac{1}{\frac{b^2}{ac}\left(\frac{w}{w_0}\right)^2 + \left(1 - \left(\frac{w}{w_0}\right)^2\right)^2}}$$

$$\frac{Rc}{F_0} = \left[\Gamma \left(\frac{\omega}{\omega_0}\right)^2 + \left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 \right]^{-1/2}$$

where $\Gamma = \dfrac{b^2}{ac}$