Trajectories

Another way we can analyze these first order systems is by looking at the trajectories of the solutions.

Trajectory = path the solution follows.

→ y as a function of x

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

→ Given in the system that I want to solve.

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$$

→ Try to solve to get trajectories.

What does this resemble?

We had talked about trajectories before for a different type of equation.

Exact $\quad \psi(x,y) = C$

$$\boxed{\psi_x(x,y) + \psi_y(x,y)\frac{dy}{dx} = 0}$$

Systems $\quad \frac{d}{dt} H(x,y) = C$

$$H_x(x,y)\frac{dx}{dt} + H_y(x,y)\frac{dy}{dt} = 0$$

This will be zero if

$$\begin{cases} \frac{dx}{dt} = -H_y(x,y) \\ \frac{dy}{dt} = H_x(x,y) \end{cases}$$

Hamiltonian System

The definition we have here is that of a Hamiltonian System.

A system is Hamiltonian if there is a function $H(x,y)$ so that

$$\begin{cases} \dfrac{dx}{dt} = -H_y(x,y) = F(x,y) \\ \dfrac{dy}{dt} = H_x(x,y) = G(x,y) \end{cases}$$

- Trajectories are $H(x,y) = c$
- Find H by integrating for exact eqns.
- Check for Hamiltonian

$$\boxed{F_x + G_y = 0}$$

$$\begin{bmatrix} 1 & 2 \\ -8 & -1 \end{bmatrix} \qquad (1-\lambda)(-1-\lambda)+16$$

$$\lambda^2+15$$

↳ Center

Example. Determine that the system

$$\frac{dx}{dt} = 2y + x \qquad \frac{dy}{dt} = -8x - y$$

is Hamiltonian find the trajectories of the system.

$$F_x + G_y = 1 - 1 = 0 \quad ✓$$

Solve

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-8x-y}{2y+x}$$

$$(2y+x)\frac{dy}{dx} = -8x-y$$

$$(8x+y) + (2y+x)\frac{dy}{dx} = 0$$

$$\psi(x,y) = \int 8x + y \, dx = 4x^2 + xy + A(y) \quad y^2$$

$$\psi_y = x + A'(y) = x + 2y$$

$$\psi(x,y) = \boxed{4x^2 + xy + y^2 = C}$$

– Ellipses.