Linearization of Non-Linear Systems

What can we say about these equilibrium solutions?

We can find them.

Autonomous Eqns – Increasing and decreasing in between.

Autonomous Systems are not quite as simple.

If $f'(y_0) > 0 \to$ unstable

$f'(y_0) < 0 \to$ asympt. stable

$$\frac{dy}{dt} = f(y)$$

$$\approx \cancel{f(y_0)}^{\,0} + f'(y_0)(y - y_0)$$

$y=0$ is unstable if $a>0$

$y=0$ is asymptotically stable if $a<0$

$$\frac{dy}{dt} = ay$$

Ce^{at}

Linearization process:

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y)$$

Assume (a,b) is an equilibrium solution

$$F(x,y) \approx \cancelto{0}{F(a,b)} + F_x(a,b)(x-a) + F_y(a,b)(y-b)$$

$$G(x,y) \approx \cancelto{0}{G(a,b)} + G_x(a,b)(x-a) + G_y(a,b)(y-b).$$

$$\frac{dx}{dt} = F(x,y) \qquad \frac{dx}{dt} = F_x(a,b)(x-a) + F_y(a,b)(y-b)$$
$$\approx$$
$$\frac{dy}{dt} = G(x,y) \qquad \frac{dy}{dt} = G_x(a,b)(x-a) + G_y(a,b)(y-b)$$

$$\Downarrow$$

$u = x-a$

$v = y-b$

$$\boxed{\begin{aligned} \frac{du}{dt} &= F_x u + F_y v \\ \frac{dv}{dt} &= G_x u + G_y v. \end{aligned}}$$

Theorem 0.1. *Take an autonomous system of the form*

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

and let (a, b) *be an equilibrium solution of this system. The* linearization *of this system near* (a, b) *is the system*

$$\frac{du}{dt} = \underline{F_x(a,b)}\, u + \underline{F_y(a,b)}\, v$$
$$\frac{dv}{dt} = \underline{G_x(a,b)}\, u + \underline{G_y(a,b)}\, v.$$

→ This linear system describes the behavior of the non-linear system near the equilibrium solution.

→ Does not describe behavior far away from the equilibrium solution.

Example. Find the linearization of the system

$$\frac{dx}{dt} = (x-2)(y-4) = F(x,y)$$
$$\frac{dy}{dt} = (x-y)(y+3) = G(x,y)$$

around the critical point (2,2)

$F_x = (y-4)$ $\quad$ $F_y = (x-2)$

$G_x = (y+3)$ $\quad$ $G_y = -1(y+3) + (x-y)1$

$F_x(2,2) = -2$ $\quad$ $F_y(2,2) = 0$

$G_x(2,2) = 5$ $\quad$ $G_y(2,2) = -5$

The linearization is
$$\frac{du}{dt} = -2u$$
$$\frac{dv}{dt} = 5u - 5v$$

$$\vec{x}\,' = \begin{bmatrix} -2 & 0 \\ 5 & -5 \end{bmatrix} \vec{x} \rightarrow \text{Nodal sink.}$$