Subspaces and Span

To get further into the idea of linear algebra, we need a few more definitions.

Definition 0.1. For a vector space V, a *subspace* W is a subset of V with the following properties:

$\mathbb{R}^2$

1. If I add two vectors in W, the sum is in W.

2. If I multiply a vector in W by a scalar, the result is in W.

Ex $V = \mathbb{R}^2$

$W = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \text{ for all real numbers } a \right\}$

W is a subspace of V.

Example. Let $V = \mathbb{R}^2$. Are the following subspaces of V?

1. $\mathbb{R}^2$ - Yes.
2. $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ - Yes. → Zero Subspace.
3. $\left\{\begin{bmatrix}a\\a\end{bmatrix} \text{ for all real numbers } a\right\}$ - Yes.
4. $\left\{\begin{bmatrix}a\\a\end{bmatrix}, \begin{bmatrix}a\\-a\end{bmatrix} \text{ for all real numbers } a\right\}$ - No!

2.
$$\begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$\alpha\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

3.
$$\begin{bmatrix}a\\a\end{bmatrix} + \begin{bmatrix}b\\b\end{bmatrix} = \begin{bmatrix}a+b\\a+b\end{bmatrix}$$
$$\alpha\begin{bmatrix}a\\a\end{bmatrix} = \begin{bmatrix}\alpha a\\\alpha a\end{bmatrix}$$

4.
$$\begin{bmatrix}a\\a\end{bmatrix} + \begin{bmatrix}b\\-b\end{bmatrix} = \begin{bmatrix}a+b\\a-b\end{bmatrix}$$
$$\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

Another example of a subspace is the *span* of a set of vectors.

Definition 0.2. If $\vec{v}_1$, ..., $\vec{v}_n$ are vectors, then a *linear combination* of these vectors is

a vector of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

for any constants $c_1, \ldots, c_n$.

Definition 0.3. Let $\vec{v}_1, \vec{v}_2, ..., \vec{v}_m$ be any set of vectors in $\mathbb{R}^n$. The *span* of these vectors

is the set of all linear combinations of these vectors. That is, it is all vectors of the form.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

for any choice of constants $c_1, \ldots, c_n$.

Ex 3 $\left\{\begin{bmatrix} a \\ a \end{bmatrix} \text{ for all } a\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Facts about Span

1. Span of a set of vectors is always a subspace.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$
$$+\ d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_n\vec{v}_n$$
$$\overline{(c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 + \cdots + (c_n+d_n)\vec{v}_n.}$$

2. In this case, the original vectors $\vec{v}_1 \ldots \vec{v}_n$ are called a <u>spanning set</u> for this subspace.

Questions about Spans

One of the most common questions to see about the span of a set of vectors is asking if a new vectors is in the span of a given set of vectors.

Example. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$?

Asking if we can find constants c_1, c_2 so that

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ -1 & 1 & 2 \\ 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 3 & 1 \end{array}\right]$$

$$\to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/3 \\ 0 & -1 & -5 \end{bmatrix}$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 5/3 \\ 0 & 0 & -10/3 \end{array}\right]$$

No Solution!

No, not in the span.

$\begin{bmatrix} 13/3 \\ 2 \\ 3 \end{bmatrix}$ is!