What is the process? Goal: Systematic Process

Aiming towards Row Echelon Form.

E1 1) Swap rows if needed to put a non-zero entry in the top left corner.

E2 2) Divide the first row by that non-zero entry to make it be 1 in that slot.

E3 3) Use the 1 to cancel out everything beneath it. Add multiples of row 1 to every other row to make their first entries zero.

4) First row and first column are fixed. Leave them alone.

5) Repeat this process on what is left of the matrix.

Example. Solve the system of equations

$$\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= 1 \\ x_1 - 4x_2 + x_3 &= 4 \\ 3x_1 + 5x_2 - 2x_3 &= 3 \end{aligned}$$

by reducing to row-echelon form.

Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & 1 \\ 1 & -4 & 1 & 4 \\ 3 & 5 & -2 & 3 \end{array}\right]$$

Swap r_1 and r_2

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 4 \\ 2 & 3 & -4 & 1 \\ 3 & 5 & -2 & 3 \end{array}\right]$$

$r_2 = r_2 - 2r_1$

$r_3 = r_3 - 3r_1$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 4 \\ 0 & 11 & -6 & -7 \\ 0 & 17 & -5 & -9 \end{array}\right]$$

$r_2/11$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 4 \\ 0 & 1 & -6/11 & -7/11 \\ 0 & 17 & -5 & -9 \end{array}\right]$$

$r_3 = r_3 - 17r_2$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 4 \\ 0 & 1 & -6/11 & -7/11 \\ 0 & 0 & -5+\frac{17\cdot 6}{11} & -9+\frac{17\cdot 7}{11} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 4 \\ 0 & 1 & -6/11 & -7/11 \\ 0 & 0 & 47/11 & 20/11 \end{array}\right]$$

$r_3 \cdot \frac{11}{47}$

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & 4 \\ 0 & 1 & -6/11 & -7/11 \\ 0 & 0 & 1 & 20/47 \end{array}\right]$$

Row Echelon Form

$x_3 = 20/47$

$x_2 - \frac{6}{11}x_3 = \frac{-7}{11} \rightarrow$

$x_2 = \frac{-7}{11} + \frac{6}{11}x_3$

$= \frac{-7}{11} + \frac{6}{11}\left(\frac{20}{47}\right) = \frac{-209}{517}$

$= -19/47$

$$x_1 - 4x_2 + x_3 = 4$$

$$x_1 = 4x_2 - x_3 + 4$$

$$= 4\left(\frac{19}{47}\right) - \left(\frac{20}{47}\right) + 4$$

$$= \frac{-76}{47} - \frac{20}{47} + \frac{188}{47} = \boxed{\frac{92}{47}}$$

Solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 92/47 \\ -19/47 \\ 20/47 \end{bmatrix}$$