There are many different spanning sets for $\mathbb{R}^2$, or any vector space/subspace.

$$\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\} \quad \left\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\} \quad \left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\} \quad \left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}3\\2\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}$$

What is different about these spanning sets?

Can always solve for the constants to hit any real vector you want.

$$\left[\begin{array}{cc|c}1 & 0 & a\\0 & 1 & b\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 3 & 2 & a\\1 & 2 & 1 & b\end{array}\right]$$

$$\left[\begin{array}{cc|c}1 & 3 & a\\2 & 1 & b\end{array}\right]$$

$$\left[\begin{array}{cc|c}1 & 0 & a\\-1 & 1 & b\end{array}\right]$$

Difference:

Linear Independence

Linear Independence of Vectors

We have discussed linear independence of solutions to differential equations before. It was discussed as two solutions being "different enough" to meet every initial condition. The definition we use here is a little bit different, but it ends up being the same once all of the details are sorted out.

Definition 0.4. We say that vectors $\vec{v}_1, \dots, \vec{v}_n$ are *linearly independent* if

the only way to satisfy

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$$

is by setting all of the c constants equal to zero.

If not, we say the vectors are linearly dependent.

Example. Are the vectors $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ linearly independent? What about $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$?

Find c_1, c_2 to make

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_1 + 0 \cdot c_2 = 0 \rightarrow c_1 = 0$$

$$-c_1 + c_2 = 0 \rightarrow c_2 = 0$$

Since 0, 0 is the only solution, these are linearly independent.

2) c_1, c_2, c_3

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These are linearly dependent!

From what we just saw, determining linear independence came down to trying to solve a system of equations, in particular, a homogeneous system of equations. This means we can use matrices and row reduction to try to work it out.

Find $c_1 \dots c_n$ so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$$

$$A\vec{c} = \vec{0}$$

$$A = \left[\vec{v}_1 \mid \vec{v}_2 \mid \dots \mid \vec{v}_n\right] \qquad \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

→ Looking for the Kernel of A.

→ Row Reduce A, and look at pivot columns.

- → Any non-pivot column means the vectors are linearly dependent.
- → I can write any non-pivot column vector as a linear combination of the pivot column vectors.

Another option: Write the vectors as rows and row reduce.

- → Operations in row reduction are linear combinations of the rows.
- → Gives a simpler way to characterize the span of the vectors.

Example. Find a linearly independent subset of

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \\ -10 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ -8 \\ 5 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 1 & 3 & 1 & 1 \\ 2 & -2 & 1 & 5 & 4 \\ -4 & 4 & -2 & -10 & -8 \\ 2 & -1 & 4 & 6 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & -6 & 1 & 9 & -2 \\ 0 & 12 & -2 & -18 & 4 \\ 0 & -5 & 4 & 10 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 19 & 15 & 4 \\ 0 & 0 & -38 & -30 & -8 \\ 0 & 0 & 19 & 15 & 4 \end{bmatrix} \xrightarrow[r_5 = r_5 - r_3]{r_4 = r_4 + 2r_3} \begin{bmatrix} \boxed{1} & 2 & 0 & -2 & 3 \\ 0 & \boxed{1} & 3 & 1 & 1 \\ 0 & 0 & \boxed{19} & 15 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Can take

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \\ 4 \end{bmatrix} \right\}$$

as a linearly independent subset.