Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors, in some sense, characterize a square matrix A. Therefore, we want to be able to compute them.

Definition 0.1. Let A be a square matrix. An *eigenvector* of A is a non-zero vector where applying A to the vector acts like scalar multiplication. That is, it is a vector $\vec{v}$ where there exists a constant λ so that

$$A\vec{v} = \lambda\vec{v}$$

λ is the eigenvalue, which has corresponding eigenvector $\vec{v}$.

It turns out, it's easier to search for eigenvalues first and then find eigenvectors. So, how can we find them?

$$A\vec{v} = \lambda\vec{v} = \lambda I\vec{v}$$

$\vec{v} = 0$ always works.

$$(A - \lambda I)\vec{v} = \vec{0}$$

If $A - \lambda I$ is invertible, then $\vec{0}$ is the only solution.

- Need $A - \lambda I$ to be not invertible.
- Look at when $\boxed{\det(A - \lambda I) = 0}$
  - → Solve for λ
  - → These solutions will be eigenvalues

What does the equation $\det(A - \lambda I) = 0$ look like in terms of λ? Let's look at a 2×2 matrix as an example.

Example. Set up the equation for $\det(A - \lambda I) = 0$ for the matrix

$$A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 3 \\ -2 & -1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (4-\lambda)(-1-\lambda) + 6$$

$$= -4 + \lambda - 4\lambda + \lambda^2 + 6$$

$$= \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

Quadratic Polynomial!

This gives us a straight-forward way to compute the eigenvalues of a matrix.

Definition 0.2. The *characteristic polynomial* of a matrix is the polynomial in λ given by

$$\det(A - \lambda I)$$

☆ Roots of this polynomial are the eigenvalues of the matrix A ☆

- If A is $n \times n$ square matrix, this will be a degree n polynomial.
  - → Quadratic has a formula
  - → Others are more complicated.