Determinants

For a square matrix A, the determinant is a number that gives a characterization of how this matrix behaves. Let's start with 2x2 matrices.

Definition 0.1. For a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant

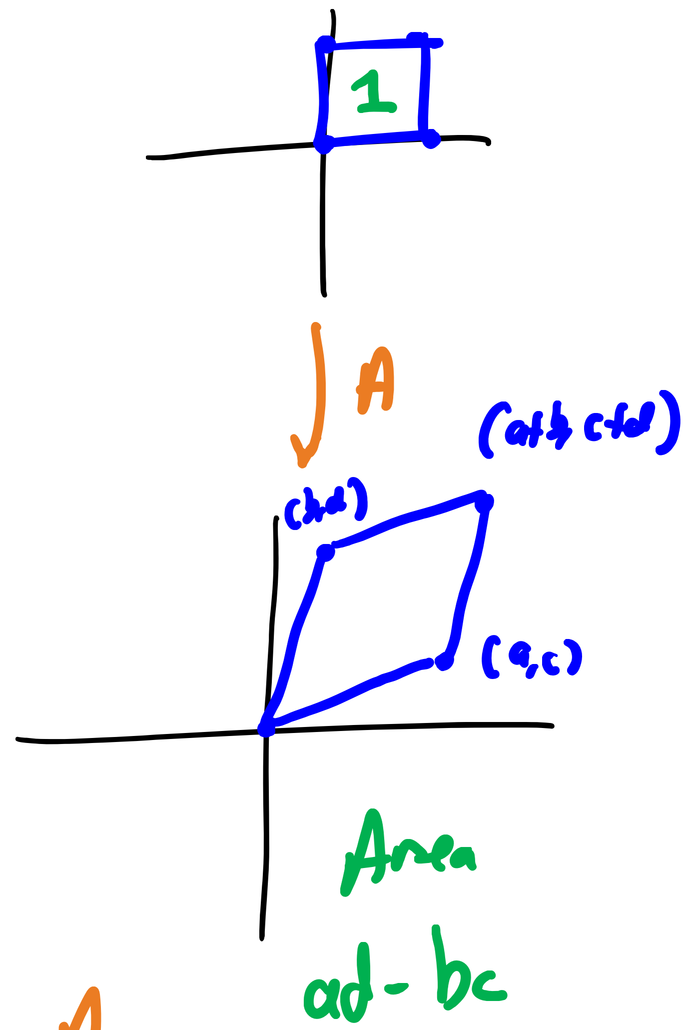
$$\det(A) = ad - bc$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$$



Determinant is how much A scales vectors.

Larger Matrices

How do we compute the determinant of bigger matrices? We do so recursively.

Definition 0.2. For an $n \times n$ square matrix A, we defined the ij *minor* of A, denoted A_{ij}, to be

the $(n-1) \times (n-1)$ matrix which is A after removing the ith row and jth column.

[$a_{ij} \to$ entries $\quad A_{ij} \to$ minors]

Example. What are A_{23} and A_{41} for the matrix

$$A = \begin{bmatrix} 1 & 0 & 5 & -1 \\ 3 & 6 & -2 & 1 \\ 2 & 3 & 0 & 1 \\ 1 & 1 & 9 & 3 \end{bmatrix}?$$

$$A_{23} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \qquad A_{41} = \begin{bmatrix} 0 & 5 & -1 \\ 6 & -2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

With this defined, we can now compute the determinant of larger matrices.

Definition 0.3. For an $n \times n$ square matrix A, the determinant $\det(A)$ is computed by choosing a row i between 1 and n, and computing

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

where a_{ij} is the entry in row i, column j of the matrix A, and A_{ij} is the ij minor of A. It can also be computed by choosing a column j between 1 and n and computing

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

2x2 "Obvious" $[a] \to \det([a]) = a$

First Row $(-1)^{i+j} a_{ij} \det(A_{ij})$

$$\begin{bmatrix} \underline{a} & \underline{b} \\ c & d \end{bmatrix}$$

$(-1)^{1+1} a \det([d])$

$+ (-1)^{1+2} b \det([c])$

$ad - bc$

→ Expansion by Cofactors.

$$\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Example. Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}.$$

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$$(-1)^{1+3}(4)\det\left(\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}\right) + (-1)^{1+4} \cdot 0 \cdot \sim + (-1)^{1+5} \cdot 0 = 4(-1-6) = \boxed{-28}$$

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$$(-1)^{1+1}(1)\det\left(\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\right) + (-1)^{1+2}(-1)\det\left(\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}\right) + (-1)^{1+3}(3)\det\left(\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}\right)$$

$$1 \cdot (2 \cdot 0 - 1 \cdot 0) + (-1)(-1)(0 \cdot 0 - 4 \cdot 1) + 1 \cdot 3(0 \cdot 0 - 4 \cdot 2)$$

$$= 0 - 4 - 24 = \boxed{-28}$$