Existence and Uniqueness

Some of the most important questions to ask about a differential equation are centered around the ideas of existence and uniqueness. These ideas are

- Existence - Is there a solution to this equation?
- Uniqueness - Is there only one solution to this differential equation?

- In the physical world, we always have these properties.

→ Our models should also have these properties.

In terms of first order equations, there are two different statements of this theorem depending on if the equation is linear or non-linear.

Theorem 0.1. *Consider the differential equation* $y' + p(t)y = g(t)$*. If the functions* $p(t)$ *and* $g(t)$ *are continuous on an interval* $I = (a, b)$ *containing the point* t_0*, then there exists a unique function* $y = \phi(t)$ *that satisfies the differential equation for each* t *in* I *that also satisfies the initial condition* $y(t_0) = y_0$ *where* y_0 *is an arbitrary prescribed initial value.*

- We have an explicit method for solving these problems.

Theorem 0.2. *Consider the differential equation* $y' = f(t, y)$*. Assume that* f *and* $\frac{\partial f}{\partial y}$ *are continuous in some rectangle* $\alpha < t < \beta$ *and* $\gamma < y < \delta$ *containing the point* (t_0, y_0)*. Then, in some interval* $t_0 - h < t < t_0 + h$*, contained inside* $\alpha < t < \beta$*, there is a unique solution of the initial value problem*

$$y' = f(t, y) \qquad y(t_0) = y_0.$$

- Much more complicated proof Technique
  - Picard Iteration.