Existence and Uniqueness

Some of the most important questions to ask about a differential equation are centered around the ideas of existence and uniqueness. These ideas are

In terms of first order equations, there are two different statements of this theorem depending on if the equation is linear or non-linear.

Theorem 0.1. *Consider the differential equation* $y' + p(t)y = g(t)$. *If the functions* $p(t)$ *and* $g(t)$ *are continuous on an interval* $I = (a, b)$ *containing the point* t_0, *then there exists a unique function* $y = \phi(t)$ *that satisfies the differential equation for each* t *in* I *that also satisfies the initial condition* $y(t_0) = y_0$ *where* y_0 *is an arbitrary prescribed initial value.*

Theorem 0.2. *Consider the differential equation* $y' = f(t, y)$. *Assume that* f *and* $\frac{\partial f}{\partial y}$ *are continuous in some rectangle* $\alpha < t < \beta$ *and* $\gamma < y < \delta$ *containing the point* (t_0, y_0). *Then, in some interval* $t_0 - h < t < t_0 + h$, *contained inside* $\alpha < t < \beta$, *there is a unique solution of the initial value problem*

$$y' = f(t, y) \qquad y(t_0) = y_0.$$