Exact Equations

Another class of equations where we can generate implicit solutions by a somewhat indirect method is "exact" equations. To discuss this method, let's start with a Calculus 3 problem.

Example. Assume that y is a function of x, and you know that x and y together satisfy the relation

$$\frac{d}{dx}\left(x^2y + 4xe^y = 8.\right)$$

Take the derivative in x of both sides of this equation to find a differential equation for $\frac{dy}{dx}$.

$$2x \cdot y + x^2 \cdot \frac{dy}{dx} + 4e^y + 4xe^y \cdot \frac{dy}{dx} = 0$$

$$\boxed{(2xy + 4e^y) + (x^2 + 4xe^y)\frac{dy}{dx} = 0}$$

$$\frac{dy}{dx} = -\frac{2xy + 4e^y}{x^2 + 4xe^y}$$

Now consider the following problem

Example. Find a solution to the differential equation

$$2xy + 4e^y + (x^2 + 4xe^y)\frac{dy}{dx} = 0$$

with initial condition $y(2) = 0$.

$$\frac{d}{dx}\left(x^2y + 4xe^y\right) = 0$$

$$x^2y + 4xe^y = C$$

$$0 + 4 \cdot 2 \cdot e^0 = C \rightarrow C = 8$$

$$\boxed{x^2y + 4xe^y = 8}$$

What is the idea here? The point is that for our given differential equation, which we had written in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

we knew that there was a function $\Psi(x,y)$ so that taking the derivative of $\Psi(x, y(x))$ in x gave us the left-hand side.

$$\frac{d}{dx}\left(\Psi(x,y(x))\right) =$$

$$\Psi(x,y)$$

Chain Rule from Multivariable Calculus

$$\frac{d}{dx}\left(\Psi(x,y(x))\right)$$

$$= \frac{\partial \Psi}{\partial x}\cdot 1 + \frac{\partial \Psi}{\partial y}\frac{dy}{dx}$$

So we need

$$\Psi_x = M, \quad \Psi_y = N$$

Definition. A differential equation $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is said to be *exact* if

there is a function $\psi(x,y)$ so that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and}$$

$$\frac{\partial \psi}{\partial y} = N(x,y)$$

In this case, we know that solutions are given implicitly by

$$\psi(x,y) = C.$$

ψ can be referred to as a potential function, stream function, or a conserved quantity.