Exact Equations

Another class of equations where we can generate implicit solutions by a somewhat indirect method is “exact” equations. To discuss this method, let’s start with a Calculus 3 problem.

Example. Assume that y is a function of x, and you know that x and y together satisfy the relation

$$x^2y + 4xe^y = 8.$$

Take the derivative in x of both sides of this equation to find a differential equation for $\frac{dy}{dx}$.

Now consider the following problem

Example. Find a solution to the differential equation

$$2xy + 4e^y + (x^2 + 4xe^y)\frac{dy}{dx} = 0$$

with initial condition $y(2) = 0$.

What is the idea here? The point is that for our given differential equation, which we had written in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

we knew that there was a function $\Psi(x,y)$ so that taking the derivative of $\Psi(x,y(x))$ in x gave us the left-hand side.

Definition. A differential equation $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ is said to be *exact* if