Solving Exact Equations

However, as of yet, this gives us no way to really determine if an equation is exact or a way to find the function Ψ that satisfies it. For the first, we have an idea from Calculus 3 again.

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$M = \psi_x \qquad N = \psi_y$$

$$M_y = \underbrace{\psi_{xy}} \qquad N_x = \underbrace{\psi_{yx}}$$

→ These must be equal

Thm If $M_y = N_x$ on some rectangle, then the equation is exact and such a ψ exists.

Now, how do we solve these equations? The idea is direct integration (basically).

Example. Find the general solution for the differential equation

$$2xy + (x^2 + 4y^3)\frac{dy}{dx} = 0$$

$M(x,y) = 2xy$

$N(x,y) = x^2 + 4y^3$

$M_y = 2x$

$N_x = 2x$

✓

This equation is exact.

Find $\psi(x,y)$ with $\frac{\partial \psi}{\partial x} = 2xy$

$\frac{\partial \psi}{\partial y} = x^2 + 4y^3$

If $\frac{\partial \psi}{\partial x} = 2xy$ so $\psi = \int 2xy\, dx$

$\frac{\partial}{\partial y}$ $\psi = x^2 y + A(y)$

$x^2 + 4y^3 = \frac{\partial \psi}{\partial y} = x^2 + A'(y) \rightarrow A'(y) = 4y^3$

$A(y) = y^4$

$$\psi(x,y) = x^2y + y^4 + C$$

Therefore the general solution is

$$\psi(x,y) = C \text{ or}$$

$$\boxed{x^2y + y^4 = C}$$