Math 244: Bifurcation Diagrams

February 16, 2021

Autonomous Equations with Parameter

- Autonomous Equation that models a physical system with something that we can change about it.

Situation: Logistic Population Growth with harvesting.

$$\frac{dy}{dt} = y(K-y) - \alpha$$

α = parameter - variable harvesting rate.

Q: How does changing α affect the equation as well as its solutions?

Definition. An *automonous equation with parameter* is a differential equation of the form

$$\frac{dy}{dt} = f_\alpha(y),$$

$f(\alpha, y)$

where f_α is a function of one variable (y) that has some dependence on α.

Examples:

$$\frac{dy}{dt} = y(y-\alpha)$$

$$\frac{dy}{dt} = y^2(y^2-\alpha)$$

$$\boxed{\frac{dy}{dt} = y(20-y)-\alpha}$$

are all examples of autonomous equations with parameter.

- For fixed α, these are standard autonomous equations

→ Can be analyzed using phase lines.

Analyzing these equations

As these are autonomous equations, we can analyze them using phase lines. The idea is to do so for given values of α.

Example. Let $f_\alpha(y) = y(y - \alpha)$. Draw a phase line for

$$\frac{dy}{dt} = f_\alpha(y)$$

for $\alpha = -2$, $\alpha = 0$, and $\alpha = 3$.

$$\frac{dy}{dt} = y(y+2)$$

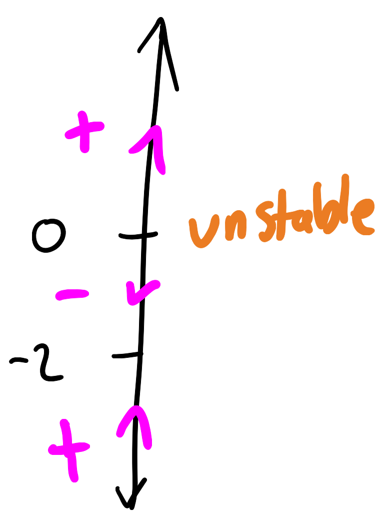
eq. sols $y=0, y=-2$

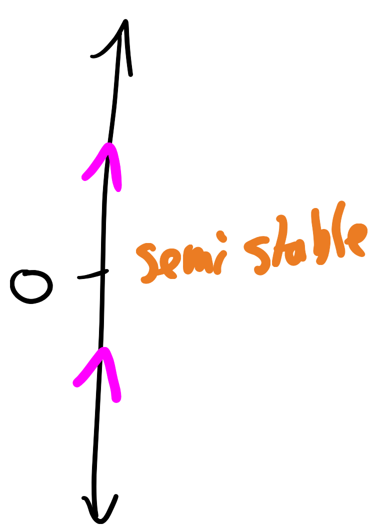
$$\frac{dy}{dt} = y^2$$

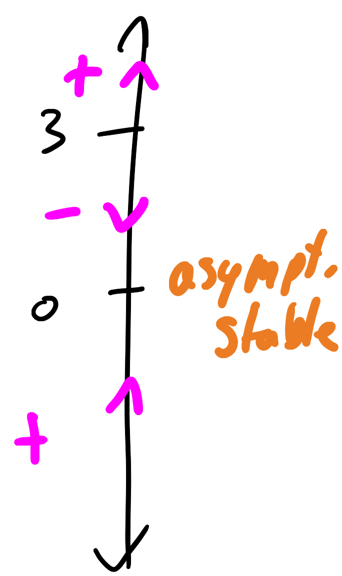
$y=0$

$$\frac{dy}{dt} = y(y-3)$$

$y=0, y=3$





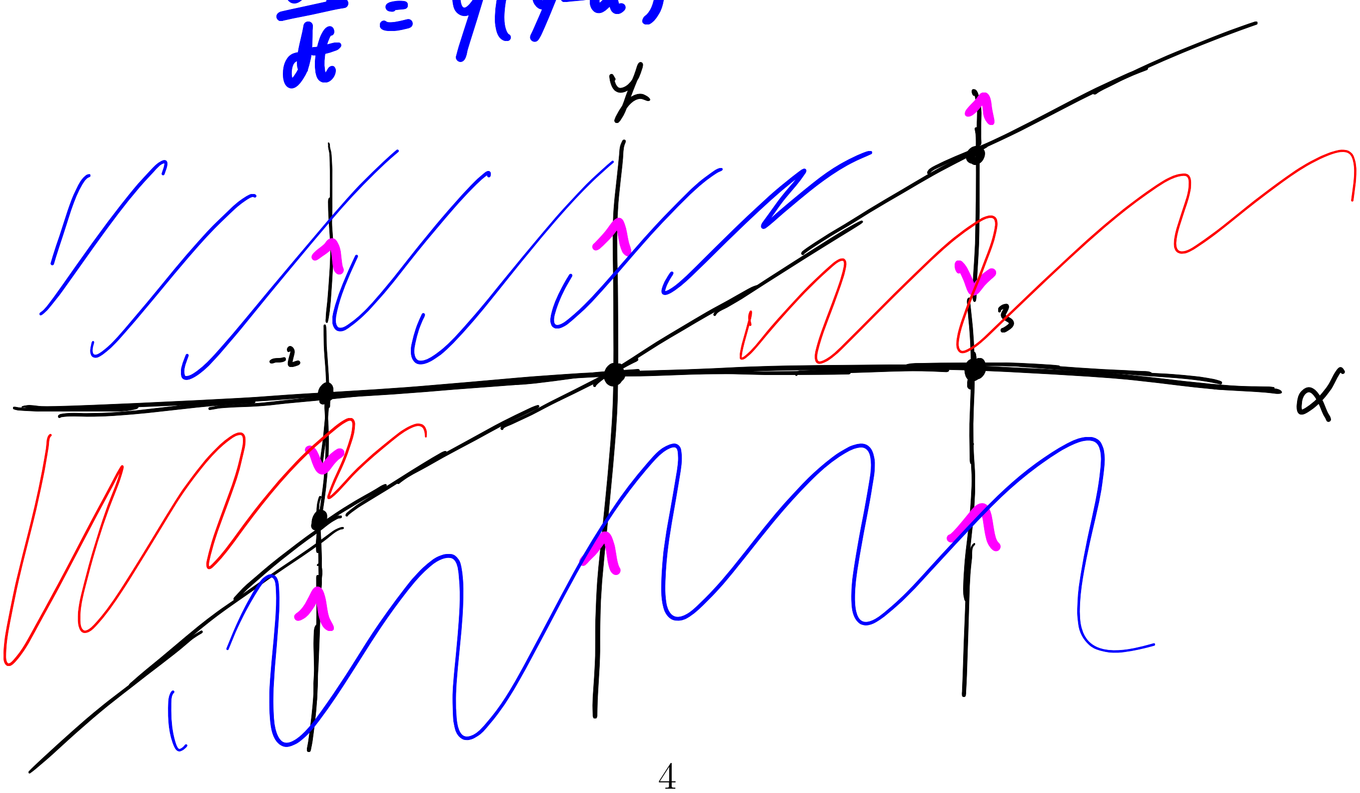


A better approach

A better way to try to analyze these types of equations is with a **bifurcation diagram**.

- Draw a graph in 2-dimensions
  - α = horizontal axis
  - y = vertical axis
- Above each α, put the phase line that corresponds to that value of α.

$$\frac{dy}{dt} = y(y-\alpha)$$



Process for Drawing Bifurcation Diagrams

1) Find equilibrium solutions
→ as a function of α if possible

2) Look at $f_\alpha(y) \sim f(\alpha, y)$. Determine where in the plane it is positive or negative.

3) Draw the graph of the equilibrium solutions
Shade the regions where it is going up or down.

Example. Sketch a bifurcation diagram for the equation

$$\frac{dy}{dt} = y(20-y) - \alpha = 0$$

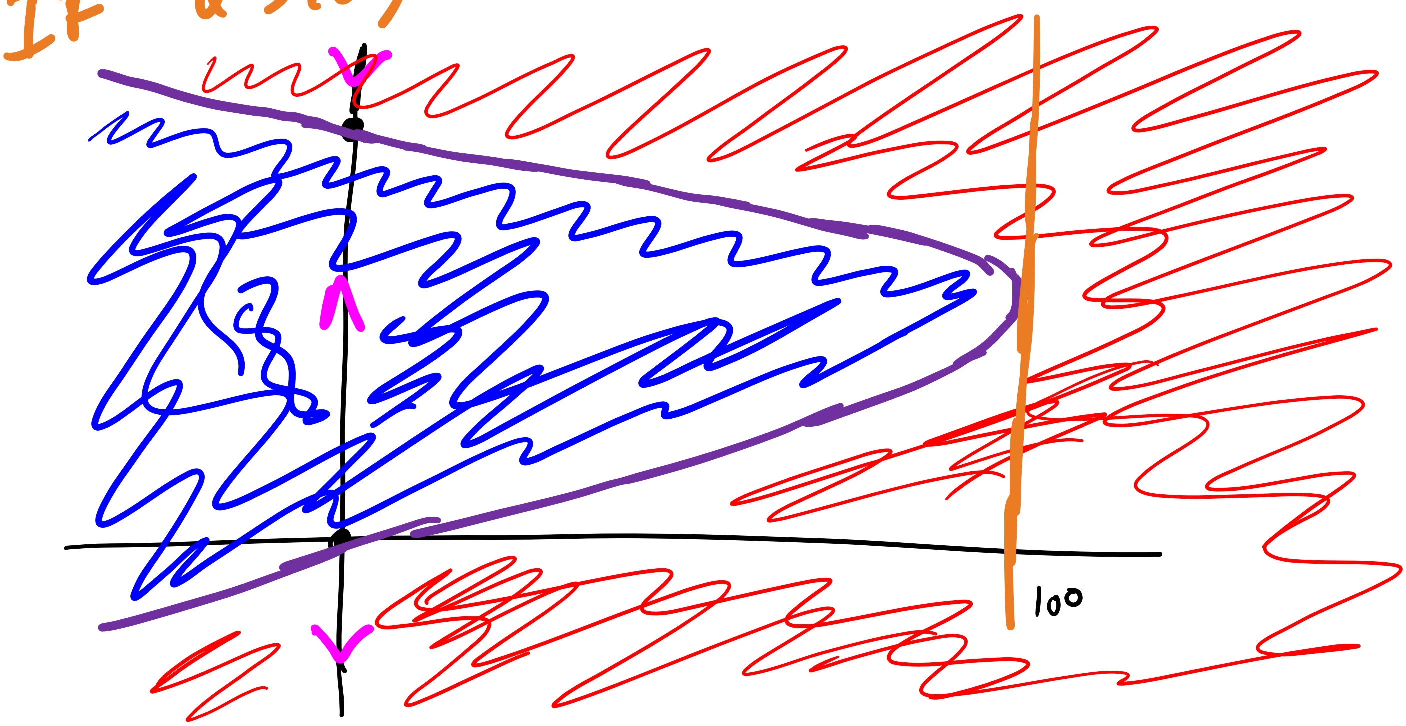
$$20y - y^2 - \alpha = 0$$

$$y^2 - 20y + \alpha = 0$$

$$y = \frac{20 \pm \sqrt{400-4\alpha}}{2} = 10 \pm \sqrt{100-\alpha}$$

If $\alpha < 100$, two real solutions

If $\alpha > 100$, no real solutions



<u>Bifurcation Point</u> - A value of α at which the qualitative behavior of the solutions change.

→ # of equilibrium solutions

→ Stability of these solutions.