Analysis Problem Sessions: Week 1

Matt Charnley

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1. (Fall 2014, Complex) Let $f(x)$ be a continuously differentiable real-valued function over $(-\infty, \infty)$ with $f(0) = 0$. Suppose that $|f'(x)| \leq |f(x)|$ for all $x \in (-\infty, \infty)$.
   (a) Show that $f(x) = 0$ for all x in a neighborhood $(-\epsilon, \epsilon)$ of 0, for some $\epsilon > 0$.
   (b) Show that $f(x) = 0$ for all $x \in (-\infty, \infty)$.

2. (Spring 2016, Real) A subset A of $\mathbb{R}^n$ is said to be *path-connected* if, given any two points $x_0, y_0 \in A$, there exists a continuous path $\phi : [0, 1] \to A$ so that $\phi(0) = x_0$ and $\phi(1) = y_0$.
   (a) Prove that if $A \subset \mathbb{R}^n$ is non-empty and path-connected, then A is connected.
   (b) Suppose now that A is an open subset of $\mathbb{R}^n$. For $x \in A$, let C_x be the set of points z in A for which there is a continuous path in A from x to z. Prove that C_x is open in A.
   (c) Continuing with the assumptions in part (b), prove that for any two points $x, y \in A$, either $C_x = C_y$, or $C_x \cap C_y = \emptyset$.
   (d) Continuing with the assumptions of (b) and (c), show that if A is connected, then A is also path-connected.

3. (Fall 2016, Real) Let n be a positive integer.
   (a) Define what it means for a set $S \subset \mathbb{R}^n$ to be connected.
   (b) Let Ω be an open, connected subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be a function such that

$$\lim_{\epsilon \to 0} \frac{f(p + \epsilon v) - f(p)}{\epsilon} = 0$$

   for every $p \in \Omega$ and $v \in \mathbb{R}^n$. Prove that f is constant in Ω.