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MATH 251: In-Class Midterm

December 3, 2015

Name: Solutions Sec: ____________

1. Let $\mathcal{D}$ be the region between the curves $y = 5 - x^2$ and $y = x^2 - 3$. Sketch the region and compute the integral of $f(x, y) = x^2$ over this region.

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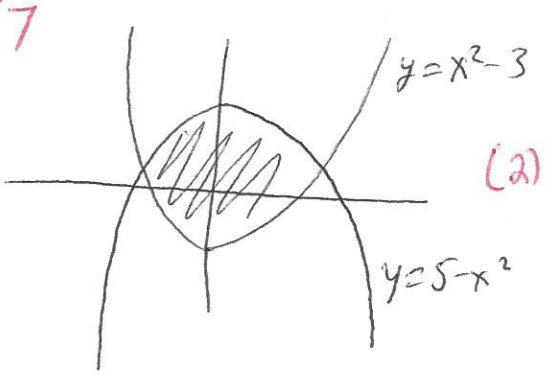

(2)

Intersections:

$5 - x^2 = x^2 - 3$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

$$\int_{-2}^{2}\int_{x^2-3}^{5-x^2} x^2 \, dy \, dx$$

$$= \int_{-2}^{2} x^2 y \Big|_{x^2-3}^{5-x^2} dx$$

(2)

$$= \int_{-2}^{2} 5x^2 - x^4 - x^4 + 3x^2 \, dx$$

$$= \int_{-2}^{2} 8x^2 - 2x^4$$

$$= \frac{8}{3}x^3 - \frac{2}{5}x^5 \Big|_{-2}^{2}$$

(2)

$$= \frac{8}{3}(2)^3 - \frac{2}{5}(2)^5 + \frac{8}{3}(+2)^3 - \frac{2}{5}(+2)^5$$

(1)

$$= \frac{16 \cdot 8}{3} - \frac{4 \cdot 32}{5} = 128\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\frac{256}{15}}$$

2. Compute the volume of the region $\mathcal{R}$ sitting above the triangle bounded by $x = 0$, $y = 0$ and $y = 1 - x$ in the xy-plane, and between the planes, $x + y + z = 5$ and $2x + y + 3z = 6$.

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$z = 5 - x - y \qquad z = \frac{1}{3}(6 - 2x - y)$

$$\int_0^1 \int_0^{1-x} \int_{\frac{1}{3}(6-2x-y)}^{5-x-y} 1 \, dz \, dy \, dx \quad (2)$$

$$= \int_0^1 \int_0^{1-x} 5 - x - y - \frac{1}{3}(6 - 2x - y) \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} 3 - \frac{1}{3}x - \frac{2}{3}y \, dy \, dx \quad (2)$$

$$= \int_0^1 3\cancel{(1-x)}y - \frac{1}{3}xy - \frac{1}{3}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 3(1-x) - \frac{1}{3}x + \frac{1}{3}x^2 - \frac{1}{3}(1-x)^2 \, dx \quad (1)$$

$$= -\frac{3}{2}(1-x)^2 - \frac{1}{6}x^2 + \frac{1}{9}x^3 + \frac{1}{9}(1-x)^3 \Big|_0^1$$

$$= \frac{1}{9} - \frac{1}{6} + \frac{3}{2} - \frac{1}{9} = 8/6 = \boxed{4/3} \quad (1)$$

$\rho = 4$

3. Find the integral of $f(x, y, z) = x + z$ over the region inside the hemisphere of radius 4 where $y \geq 0$, and above the plane $z = 2$.

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$0 \leq \theta \leq \pi$

$z = 2 \rightarrow \rho \cos\varphi = 2$

$\rho = 2/\cos\varphi$

$4 = \rho = 2/\cos\varphi$

$\cos\varphi = 1/2$ (2)

$\varphi = \pi/3$.

$$\rightarrow \int_0^{\pi/3}\int_0^{\pi}\int_{2/\cos\varphi}^{4} (\rho\cos\theta\sin\varphi + \rho\cos\varphi)\rho^2\sin\varphi \, d\rho \, d\theta \, d\varphi$$ (2)

$$= \int_0^{\pi/3}\int_0^{\pi}\int_{2/\cos\varphi}^{4} \rho^3\cos\theta\sin^2\varphi + \rho^3\cos\varphi\sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{\pi/3}\int_0^{\pi} \frac{\rho^4}{4}\left(\cos\theta\sin^2\varphi + \cos\varphi\sin\varphi\right)\Big|_{2/\cos\varphi}^{4} d\theta \, d\varphi$$

$$= \int_0^{\pi/3}\int_0^{\pi}\left(64 - \frac{4}{\cos^4\varphi}\right)\left(\cos\theta\sin^2\varphi + \cos\varphi\sin\varphi\right) d\theta \, d\varphi$$ (1)

$$= \int_0^{\pi/3}\left(64 - \frac{4}{\cos^4\varphi}\right)\left(\sin\theta\sin^2\varphi + \theta\cos\varphi\sin\varphi\right)\Big|_0^{\pi} d\varphi$$ (1)

$$= \pi\int_0^{\pi/3} 64\cos\varphi\sin\varphi - 4\frac{\sin\varphi}{\cos^3\varphi} \, d\varphi$$

$u = \cos\varphi$

$du = -\sin\varphi \, d\varphi$

$$= \pi\left[-32\cos^2\varphi - 2\cos^{-2}\varphi\right]_0^{\pi/3} = \pi\left[-\frac{32}{4} - 2\cdot 4 + 32 + 2\right]$$

$$= \boxed{18\pi}$$ (1)