Hint: All critical and inflection points are Integers.

MATH 135: Quiz 9

November 4, 2014

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Name: Solutions Sec: ____________

Fill in the following table and use it to sketch the graph of the function $f(x)$ below.

$$f(x) = \frac{(x+2)(x-2)^2}{x} \qquad f'(x) = \frac{2(x-2)\overbrace{(x^2+x+2)}^{A}}{x^2} \qquad f''(x) = \frac{2(x+2)\overbrace{(x^2-2x+4)}^{B}}{x^3}$$

For each row in the table, list the interval(s) or point(s) where f has the given property. If none exist, write "none". The axes for the sketch are on the back of this page.

Roots ($f(x) = 0$)	$x = 2,\ x = -2$	(1)
Increasing	$(2, \infty)$	(1)
Decreasing	$(-\infty, 0) \cup (0, 2)$	
Concave Up	$(-\infty, -2) \cup (0, \infty)$	(1)
Concave Down	$(-2, 0)$	
Critical Points	$x = 2$ (Minimum)	(1)
Inflection Points	$x = -2$	
Horizontal Asymptotes	None	(1)
Vertical Asymptotes	$x = 0$	

A, B do not factor, so by the hint they are never 0.

By plugging in 0, we see that $A > 0$, $B > 0$ for all x

→ You can also plug in points to check this in each case (1)

f: $\frac{(-)(-)^2}{(-)} = +$ | -2 | $\frac{(+)(-)^2}{(-)} = -$ | 0 | $+$ | 2 | $+$

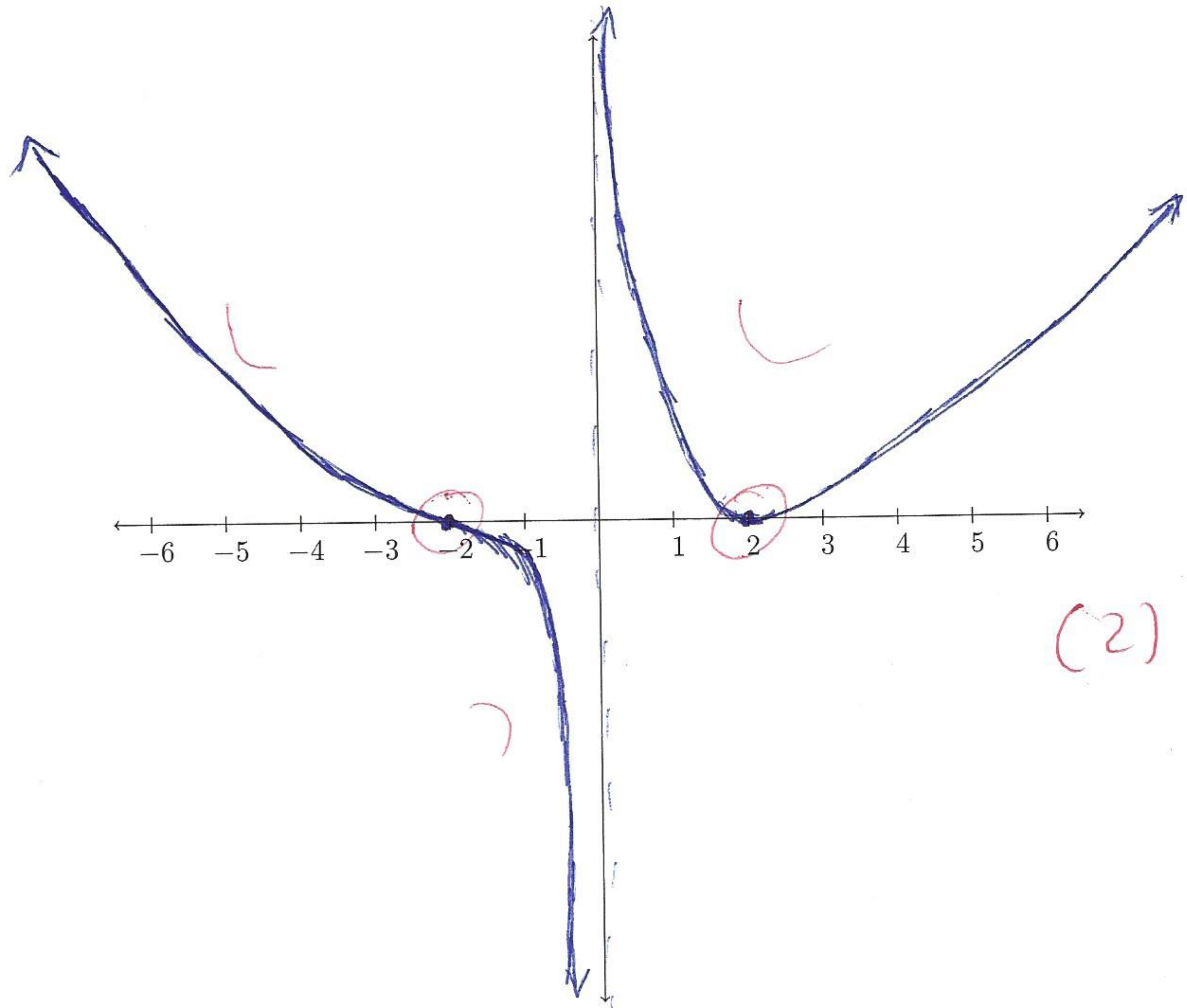
or $\lim_{x \to 0^-} \frac{(x+2)(x-2)^2}{x} = -\infty$

$\lim_{x \to 0^+} \frac{(x+2)(x-2)^2}{x} = +\infty$.

f': $-$ | 0 | $-$ | 2 | $+$ (1)

f'': $+$ | -2 | $-$ | 0 | $+$ (1)

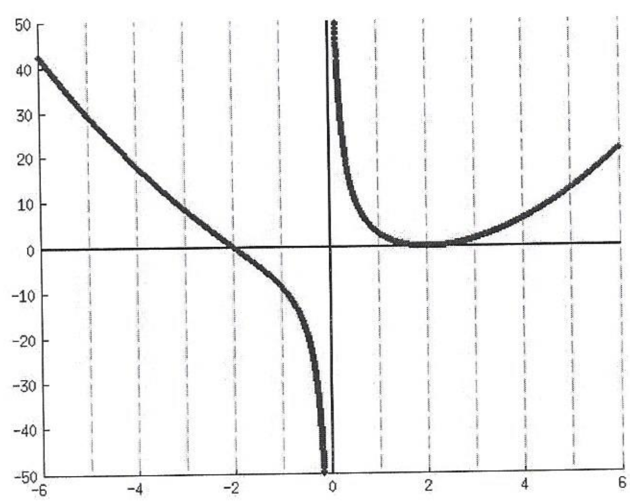
1



(2)

Asymptote

Actual Graph:



Quiz 9 Detailed Write-up

$$f(x) = \frac{(x+2)(x-2)^2}{x} \qquad f'(x) = \frac{2(x-2)(x^2+x+2)}{x^2} \qquad f''(x) = \frac{2(x+2)(x^2-2x+4)}{x^3}$$

Hints: 1. Roots = X - intercepts

2. All critical points are integers

3. $x^2+x+2>0 \qquad x^2-2x+4>0$ all x

and they don't factor.

• Roots / X - intercepts: Set $f(x)=0$ and solve.

$$0 = \frac{(x+2)(x-2)^2}{x}$$

$$0 = (x+2)(x-2)^2 \quad \leftarrow \text{ Numerator must be zero.}$$

$$\boxed{x=-2, \quad x=+2}$$

So these are the roots. ✓

• Vertical Asymptotes - Look where f is undefined.

• The only place f is undefined is at $x=0$.

• As $x \to 0$, the numerator goes to $(0-2)(0+2)^2 = 8$ and the denominator goes to zero.

• Therefore, this is a vertical asymptote, and the only one since f is defined everywhere else.

$\boxed{\text{Vertical Asymptote: } x=0}$

- <u>Horizontal Asymptotes</u> – $\lim_{x\to\infty} f(x)$

Method 1) The numerator of f goes like x^3, while the denominator is just x.

Therefore, there is no asymptote.

Method 2) Calculate the limit.

$$\lim_{x\to\infty} \frac{(x+2)(x-2)^2}{x} = \lim_{x\to\infty} \frac{x^3-2x^2-4x+8}{x} \cdot \frac{(1/x^3)}{(1/x^3)}$$

$$= \lim_{x\to 0} \frac{1-2/x-4/x^2+8/x^3}{1/x^2}$$

Taking the limit of each term

$$\text{"="} \quad \frac{1-0-0+0}{0} = \infty \text{ or undef.}$$

Since this limit is undefined / ∞, there is no horizontal Asymptote.

$\boxed{\text{Horizontal Asymptote: None}}$

- Critical Points
  - 1^{st} order: $f'(x) = 0$ and solve

$$0 = \frac{2(x-2)(x^2+x+2)}{x^2}$$

$$0 = 2(x-2)(x^2+x+2)$$ Numerator must be zero

$x - 2 = 0$ or $x^2 + x + 2 = 0$

$x = 2$ NO SOLUTIONS

By the hints given in class

$$\begin{cases} x^2+x+2 > 0 \quad \text{all } x \text{ ; or} \\ \text{All critical points are integers + } x^2+x+2 \text{ does not factor} \end{cases}$$

We get that x^2+x+2 is Never 0.

→ Quadratic Formula can also check that.

Therefore, the only 1^{st} order critical point is at $x = 2$ (and the asymptote at $x=0$)

$$\boxed{\text{Critical Points} \quad x = 2}$$

- 2^{nd} Order : $f''(x) = 0$

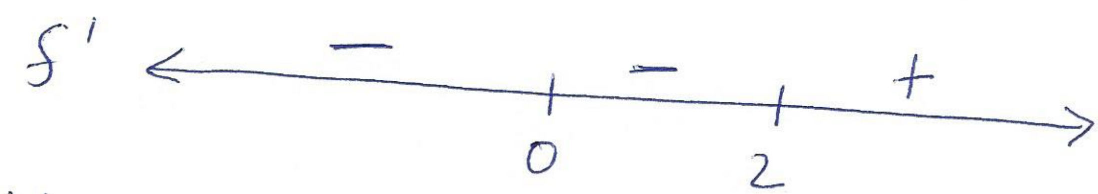
$$0 = \frac{2(x+2)(x^2-2x+4)}{x^3}$$

$$0 = 2(x+2)(x^2-2x+4)$$

$x = -2$ NO SOLUTIONS by the Same argument for 1^{st} order.

$\boxed{2^{nd} \text{ Order (possible inflection Point) } x = -2}$

- Increasing / Decreasing : Check f' in intervals
  - We need to mark the asymptote and critical points.



- You can either do this by positive/negative arguments, or plugging in points.

$$f'(-1) = \frac{2(-1-2)((-1)^2-1+2)}{(-1)^2} = \frac{2(-3)(1-1+2)}{1} = -12$$

$$f'(1) = \frac{2(1-2)(1^2+1+2)}{1^2} = -8$$

$$f'(3) = \frac{2(3-2)(3^2+3+2)}{3^2} = \frac{2 \cdot 1 \cdot 14}{9} = 28/9$$

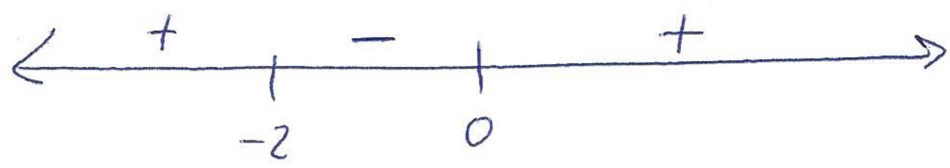
$\boxed{\text{and: } x = 2 \text{ is a minimum}}$

$\boxed{\text{Increasing: } (2, \infty) \quad \text{Decreasing: } (-\infty, 0) \cup (0, 2).}$

- Concave Up / Down : Check f'' on intervals

  - Again mark the 2^{nd} order critical point(s) and asymptote.

f''



$$f''(-3) = \frac{2(-3+2)((-3)^2-2(-3)+4)}{(-3)^3} = \frac{2\cdot(-1)\cdot(9+6+4)}{-27} = \frac{38}{27}$$

$$f''(-1) = \frac{2(-1+2)((-1)^2-2(-1)+4)}{(-1)^3} = \frac{2(1)(1+2+4)}{-1} = -14$$

$$f''(1) = \frac{2(1+2)(1^2-2+4)}{1^3} = \frac{2\cdot 3\cdot 3}{1} = 18$$

$\boxed{\begin{array}{l}\text{Concave up : } (-\infty,-2) \cup (0,\infty) \\ \text{Concave down : } (-2,0)\end{array}}$

and $\boxed{x=-2 \text{ is an inflection point}}$

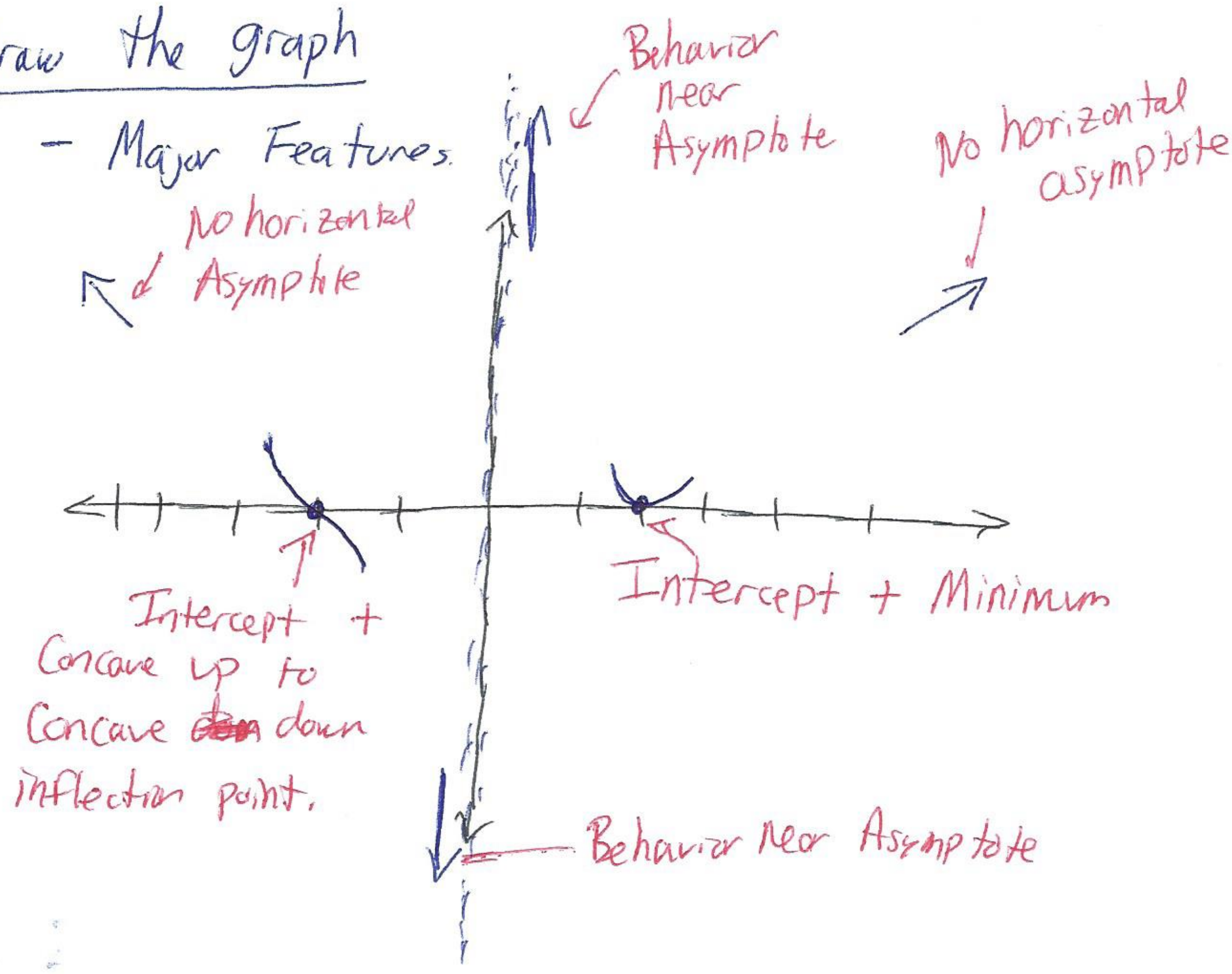
- Behavior near the asymptote

  - As $x \to 0^-$, $f'(x) < 0$, so the graph must go down to $-\infty$ on the left side

  - As $x \to 0^+$, $f'(x) < 0$, so the graph must come down from $+\infty$ on the right.

• Draw the graph

- Major Features.



- Fill in the curve.