

Math 251: Computational Lab 5 Background

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1 Area/Volume and Mass

If we have a region D of either 2 or 3 dimensional space, we can find the area (in two dimensions) or volume (in 3 dimensions) by integrating 1 over the region. For example, the area between $x = -1$ and $x = 1$ between the graphs of $y = x^2$ and $y = 2x - 5$ can be calculated by

$$\int_{-1}^1 \int_{2x-5}^{x^2} 1 \, dy \, dx.$$

If one is given a density for the object, that is, a mass per unit area (in two dimensions) or mass per unit volume (in three dimensions), one can find the total mass of the object by integrating the density over the region. That is, if the density of the unit cube $0 \leq x, y, z, \leq 1$ is given by the function $\rho(x, y, z)$, the total mass of this cube can be computed by

$$\int_0^1 \int_0^1 \int_0^1 \rho(x, y, z) \, dx \, dy \, dz.$$

If the region is given in a way where it makes sense to use a different coordinate system, all of this can be done in the same way, but you will need to convert the density into the appropriate coordinate system and add in the Jacobian factor. If we were to use the same density ρ over the unit sphere, we could do this by

$$\int_0^1 \int_0^{2\pi} \int_0^\pi \rho(r \cos(\theta) \sin(\phi), r \sin(\theta) \sin(\phi), r \cos(\phi)) r^2 \sin(\phi) \, d\phi \, d\theta \, dr$$

and you could do similar calculations in cylindrical coordinates. Whenever you are trying to carry out these calculations in a computer algebra system, you should be aware of the fact that none of them know how to handle spherical and cylindrical coordinates directly. In order to compute these integrals, you'll need to manually convert them (on paper is the easiest way) and write the integrals in polar coordinates where you multiply the desired function by the Jacobian from the change of variables.

2 Center of Mass

The other part of this lab introduces the idea of the center of mass of an object. From a physical point of view, this is the point at which one could hang the object and it would balance evenly. One can calculate the center of mass by finding the moments in each of the coordinate directions and dividing each of them by the total mass. To compute the x -moment, we take the integral that we use to find the total mass, multiply the function on the inside by x , and carry out the same integral. That is, for our unit cube from before, the total mass is given by

$$\int_0^1 \int_0^1 \int_0^1 \rho(x, y, z) \, dx \, dy \, dz$$

and we can find the x -moment by

$$\int_0^1 \int_0^1 \int_0^1 x\rho(x, y, z) dx dy dz.$$

The same goes for each of the other coordinate directions by changing the thing in front of the density to y or z respectively. **Remember:** if you want to change this integral into different coordinates, do so after multiplying by x , y , or z and add the Jacobian factor in as usual. To then find the center of mass x -coordinate, we divide the x -moment by the total mass. That is

$$COM_x = \frac{\int_0^1 \int_0^1 \int_0^1 x\rho(x, y, z) dx dy dz}{\int_0^1 \int_0^1 \int_0^1 \rho(x, y, z) dx dy dz}.$$

You can do the same for each of the y and z coordinates to find the full center of mass of the object.