# Math 251: Computational Lab 4 Background

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## 1 Optimization

Calculus 1 tells us that any function (of one variable) on a closed interval attains its maximum and minimum value. Furthermore, it tells us how to find these points:

1. Look for critical points (where the derivative is zero or does not exist),
2. Check the value of the function at these points and at the endpoints of the interval, and
3. Find the maximum and minimum from this list.

The point of using Calculus for this approach is it tells us that critical points and end points are the only places we need to check in order to find the largest and smallest values of the function. For functions of multiple variables, the idea is the same, but it becomes more complicated because the "endpoints" part of the process is more than just two points.

If we want to find the maximum and minimum of a function $f$ on a domain $D$ (for this problem, $D$ is the set of points where $g \leq 1$) we need to look for all possible interior critical points and then look on the boundary. For interior points, the process for looking for critical points is similar. Instead of the derivative being zero, we need to look for points where the gradient $\nabla f$ is either zero (as a vector) or doesn't exist. For the functions in this problem, the gradient will exist everywhere, so you only need to check for points where it is zero. Since $f$ is a function of 4 variables, looking for critical points involves finding points $(x, y, z, w)$ so that

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial z} = 0 \qquad \frac{\partial f}{\partial w} = 0.$$

**Note:** Because of the simplicity of these problems, it is possible for the critical points to not be isolated, i.e., there could be a curve of points where the gradient is zero. For instance, if we take the function

$$f(x, y, z, w) = 2y + wz + 2.0y^2$$

then the gradient of $f$ will be

$$\nabla f = (0, 2 + 4y, w, z)$$

which vanishes for $w = 0$, $z = 0$, $y = -0.5$ and *any* value of $x$. It turns out that for this to happen, the function $f$ must not depend on (in this case) $x$, and so the value of $f$ will be the same at all of these critical points. Depending on your numerical system, you may get an expression that showcases this fact, or it may just give you a particular point on the line. If it gives you a point, it doesn't hurt to check it normally, but if it gives you a line (or undefined variables), you can ignore that in your solution. Since $f$ is constant along that line, you will pick up these values when the curve in question crosses the surface $g = 1$ in the Lagrange multiplier part of the problem.

After finding these points, we then need to make sure that it's inside $D$. If it isn't inside $D$, then we won't want to include it in our potential critical points. For this particular problem, it involves checking to see if $g$ at this point where the gradient of $f$ vanishes is less than 1.

Once we have checked this point, we then have to look at the boundary of $D$. For the one dimensional case, this is just two endpoints, but it is something more complicated for multiple dimensions. The easiest way to handle this part of the problem is Lagrange Multipliers.

## 2 Lagrange Multipliers

Lagrange multipliers can be used to find the maximum or minimum values of a function $f$ with respect to a constraint $g = c$ for another function $g$. For our particular problem, we want to use $g = 1$ as our constraint. The method here says that to find this maximum or minimum value, we need to look for points where the gradient of $f$ is parallel to the gradient of $g$, that is, solutions to the equation

$$\nabla f = \lambda \nabla g$$

for some number $\lambda$.
This approach makes sense for two reasons

- $\nabla f$ points in the direction of steepest ascent of the function $f$.
- $\nabla g$ points perpendicular to the surface $g = 1$.

Thus, if $\nabla f$ is parallel to $\nabla g$, there is no way to move on the surface $g = 1$ that will be in the direction of $\nabla f$, so that the function $f$ will not strictly increase or decrease in these directions. In order to find the points of this from, we need to solve a system of 5 equations for 5 unknowns ($x$, $y$, $z$, $w$, $\lambda$):

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \qquad \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \qquad \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \qquad \frac{\partial f}{\partial w} = \lambda \frac{\partial g}{\partial w} \qquad g = 1.$$

Once we get the set of solutions to these equations, making sure that $\lambda$ is real valued, we then need to plug them all into the function $f$ and see which is the largest and smallest. For the final check for the maximum and minimum, we also need to include the critical points from the first part of the problem, provided that point is within the domain where $g \leq 1$.