Math 251: Computational Lab 3 Background

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1 Curves and Surfaces

There are several ways that we can define curves (in $\mathbb{R}^2$) and surfaces (in $\mathbb{R}^3$). One of them is parametrically, like how we used vector-valued functions to define curves in $\mathbb{R}^3$ in the previous lab. The same can be done for surfaces, and you will see that later in the course. However, another way to do this is by implicitly defining curves and surfaces as the zero-set of a function. For example, the line $y = 2x+5$ in $\mathbb{R}^2$ can be written as the set of points (x, y) where $f(x, y) = 0$ for the function $f(x, y) = 2x + 5 - y$. Depending on the curve, it may be easier to write it in the form $f(x, y) = 0$ or to write it parametrically. For this lab, we will be focusing on curves defined by $f(x, y) = 0$.

The fact that this gives you a curve (or multiple curves) is the fact that we have two variables, and one equation, which leaves one free variable. Therefore, the resulting points should be one-dimensional, which is a curve. The same goes for one dimension higher. If we have a function $g(x, y, z) = 0$, this should give us a two-dimensional object, which is a surface.

2 Tangent Lines and Planes

For parametric definitions of curves, finding tangent lines is fairly easy: just take a derivative, that gives you a tangent vector, and then you can use that and the point on the curve to define a line. We want to be able to do the same for curves or surfaces that are implicitly defined as $f(x, y) = 0$ or $g(x, y, z) = 0$. The key point to realizing this is that the function f or g must be constant along the curve or surface respectively. This means that if I take the directional derivative of f or g in any direction that is tangential to the curve or surface, I must get zero. So, by the formula for the directional derivative, we have that

$$\nabla f \cdot \tau = 0$$

for any tangential direction τ. Thus, ∇f must be perpendicular to all tangential directions at a given point. What else satisfies this condition? The normal vector to the tangent plane. Therefore, the gradient of f at a point p must be parallel to the normal vector to the tangent plane to the surface at p. This means we can use $\nabla f(p)$ as the normal vector $\overrightarrow{n}$ to define the tangent plane to the surface $f(x, y, z) = 0$.

In summary, in order to find the tangent plane to a surface $f(x, y, z) = 0$ at a point $p = (a, b, c)$ on the surface, we can do the following:

1. Compute ∇f and evaluate it at p. Call this vector $\overrightarrow{n} = \langle n_1, n_2, n_3 \rangle$.
2. Calculate $d = n_1 a + n_2 b + n_3 c = \overrightarrow{n} \cdot p$.
3. The tangent plane is then given by the equation

   $$n_1 x + n_2 y + n_3 z = d.$$

   You could also write this as

   $$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0.$$

The exact same process works for tangent lines in $\mathbb{R}^2$, the only changes are that there are only two variables (x and y), but just dropping the z and c terms from the process above gives you the procedure for 2 dimensions.