

Math 251: Computational Lab 2 Background

August 23, 2019

1 Vector-Valued Functions and Curves

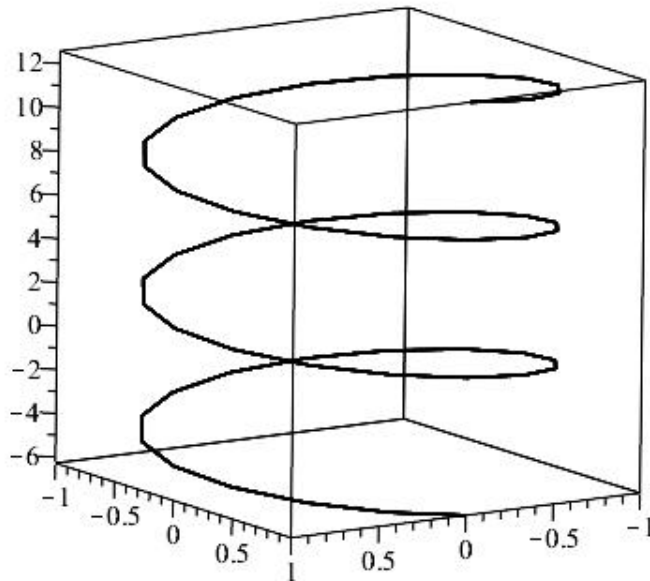
In class, you have talked about vector-valued functions, that is, functions of the form

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

where the value at every time t is a vector (instead of a single number in the case of most functions you've seen before $f(x)$). You can think of these functions as plotting points in 3-dimensional space as a function of time, which traces out a curve in space. For instance, the function

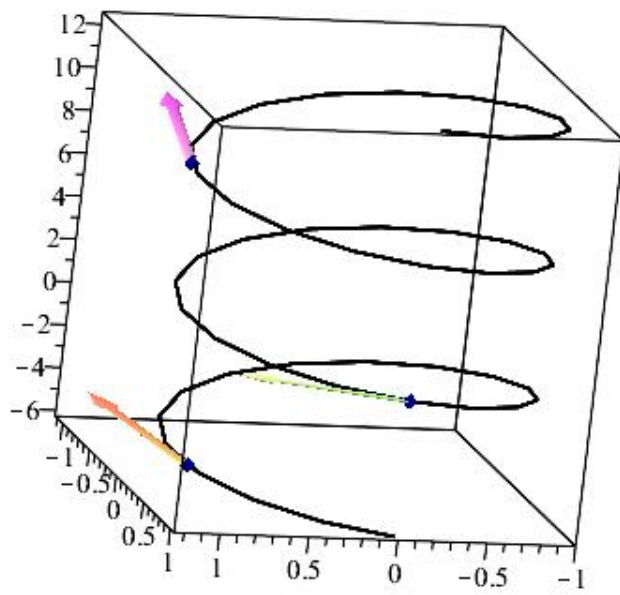
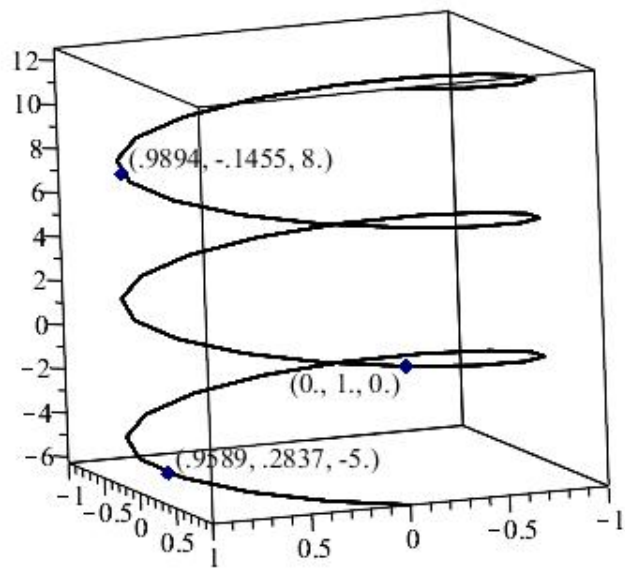
$$\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$$

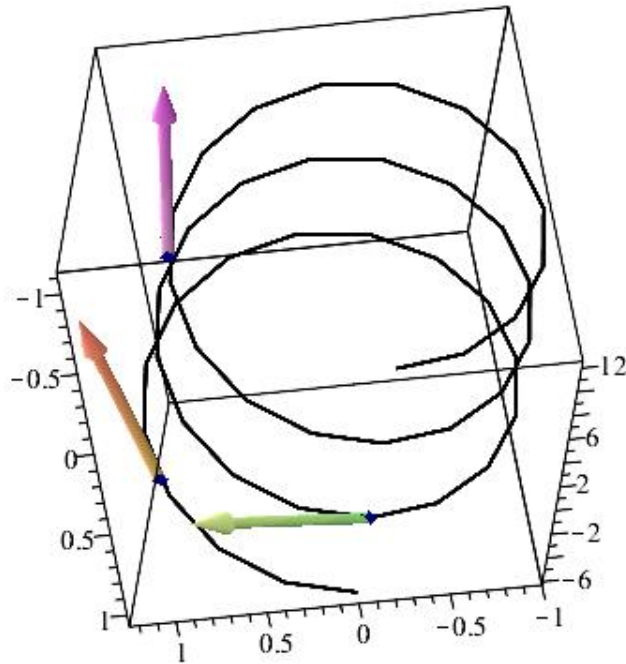
traces out the helix given in the picture below.



This curve is plotted from $t = -2\pi$ to $t = 4\pi$. Every t value in this range corresponds to a point on this curve; for instance, the value at $t = 0$ is the point $(0, 1, 0)$. The plot below shows several different points on this curve labelled with their corresponding t values. These points are also marked with an arrow indicating $\vec{r}'(t)$.

As you can notice, the vector $\vec{r}'(t)$ is a tangent vector to the curve $r(t)$; it indicates the direction the curve is travelling at that point. However, if you want a unit tangent vector to the curve, you would need to divide this vector by its length, as this will likely not be 1 for a given function $\vec{r}(t)$.





These vector-valued functions can be thought of as a particle moving around in space. This gives a physical interpretation to various mathematical quantities that can be computed from the function $\vec{r}(t)$.

1. $\vec{r}(t)$ is the *position* of the particle
2. $\vec{r}'(t)$ is the *velocity* of the particle
3. $\|\vec{r}'(t)\|$ is the *speed* of the particle

2 Arc Length

One of the main things we want to calculate for these situations is the arc-length of a curve, which is the distance travelled in terms of the motion of the particle. The key thing to realize here is that we don't want the total displacement, i.e., how far we moved from our initial position, but the distance travelled. The point here is that we want the length of a curve, i.e., a circle should not have arc length zero because you end at the same place you start.

If you think about driving in a car, you can calculate total displacement by multiplying the car speed by how long you were travelling at that speed. Even if your path winds around, you get total distance travelled by multiplying speed by the time spent at that speed. With the same motivation, we can calculate the arc length of a curve from $t = a$ to $t = b$ by

$$\int_a^b \|\vec{r}'(t)\| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

This is also just referred to as the *length* of the curve between these two points.

3 Arc Length Parametrization

For a given curve (the graph of the vector valued function in three dimensions) there are many different ways to write a function that traces it out. For instance, for the helix plot drawn earlier, the functions

1. $\vec{r}_1(t) = \langle \sin(t), \cos(t), t \rangle \quad -2\pi \leq t \leq 4\pi$
2. $\vec{r}_2(t) = \langle \sin(2t), \cos(2t), 2t \rangle \quad -\pi \leq t \leq 2\pi$
3. $\vec{r}_3(t) = \langle \sin(t/4), \cos(t/4), t/4 \rangle \quad -8\pi \leq t \leq 16\pi$
4. $\vec{r}_4(t) = \langle \sin(t^3), \cos(t^3), t^3 \rangle \quad (-2\pi)^{1/3} \leq t \leq (4\pi)^{1/3}$

all trace out the exact same curve in \mathbb{R}^3 . An inspection of these functions reveals that they are all basically related by a change of variables, replacing t by a different function ($2t$, $t/4$, or t^3 , respectively). From a mathematical point of view, we want to be able to say things about curves, not just vector-valued functions. Thus, we need to pick a particular parametrization of the curve to be able to use calculus to analyze it. This is where arc length parametrization comes in. For a curve, we say that an *arc-length parametrization* is a vector-valued function $\vec{r}_a(s)$ (s is usually used as the independent variable here) that traces out the curve and has $\|\vec{r}'_a(s)\| = 1$ for all values of s . For example, the parametrization

$$\vec{r}_a(s) = \left\langle \sin \frac{s}{\sqrt{2}}, \cos \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle \quad -2\sqrt{2}\pi \leq s \leq 4\sqrt{2}\pi$$

is an arc-length parametrization of the helix graph plotted earlier. If you calculate $\vec{r}'_a(s)$ and take the magnitude, you will get 1 for all values of s .

So, if we're given a function $\vec{r}(t)$, how do we find an arc-length parametrization of the curve corresponding to this function? Our goal is to find a change of variables so that this will work, i.e., find a function $g(s)$ so that setting $t = g(s)$, we have that

$$\vec{r}_a(s) = \vec{r}(g(s))$$

is an arc-length parametrization. If we apply the chain rule to the right-hand side of that expression, we see that

$$\vec{r}'_a(s) = \frac{d}{ds} \vec{r}(g(s)) = \vec{r}'(g(s))g'(s) = \vec{r}'(t)g'(s)$$

and since we want $\|\vec{r}'_a(s)\| = 1$, we have that

$$1 = g'(s)\|\vec{r}'(t)\| \quad \text{or} \quad g'(s) = \frac{1}{\|\vec{r}'(t)\|}.$$

Now, there's a lot of math that goes into finding a function g with that particular derivative. The way we do it is as follows.

1. Compute the function

$$L(t) = \int_a^t \|\vec{r}'(\tau)\| d\tau$$

where a can be any t -value on your curve, so you might as well pick the starting value. This computes the length of the curve up to time t . Think about what $L'(t)$ is. (Hint: Fundamental Theorem of Calculus).

2. Set $g(s)$ to be the inverse function to L , i.e., $L(g(s)) = s$ and $g(L(t)) = t$. More directly, this can be done by taking the expression $s = L(t)$ and solving out for t to give $t = g(s)$.

The function g constructed this way will have the appropriate derivative, and can be used for forming the arc-length parametrization of the curve. In most cases, the algebra required to do these calculations is very complicated, which is why you will be doing it for a computational lab instead of doing a bunch of hand-written homework assignments on it.