Math 251: Computational Lab 1 Background

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1 Vectors

A vector in $\mathbb{R}^3$ is given by two points P and Q in $\mathbb{R}^3$, $\overrightarrow{v} = \overrightarrow{PQ}$, and is drawn as an arrow from P to Q. We can also think of vectors as triples of the form

$$\langle a, b, c \rangle$$

where a, b, and c are the x, y, and z components of the vector. With this formulation, we ignore the particular base point and say that vectors are equivalent if they have the same x, y and z coordinates. For both points and vectors, the subscripts 1, 2, and 3 will denote the x, y, and z components respectively.

For a given pair of points P and Q, we can find the vector between them by subtracting the individual coordinates:

$$\overrightarrow{v} = \overrightarrow{PQ} = Q - P = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle.$$

The length of a vector v is given by

$$\overrightarrow{v} = \langle a, b, c \rangle \qquad ||\overrightarrow{v}|| = \sqrt{a^2 + b^2 + c^2}.$$

2 Dot Product

The dot product of two vectors generally looks at how much two vectors point in the same direction. If two vectors have a dot product of zero, we say that they are orthogonal, i.e., they point in entirely different directions. The dot product of vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ is computed by

$$\overrightarrow{v} \cdot \overrightarrow{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

The angle θ between two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ can be found by

$$\cos\theta = \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{||\overrightarrow{v}||||\overrightarrow{w}||}$$

where θ should be between 0 and π.

3 Cross Product

Physically, the cross product can be most easily seen in the concept of torque

$$\overrightarrow{T} = \overrightarrow{F} \times \overrightarrow{r}$$

where $\overrightarrow{F}$ is the applied force and $\overrightarrow{r}$ is the vector between the location of the force and the rotation point. Applying the same force farther away should increase the torque, and applying the force perpendicular to the radius would have the largest effect. In addition, applying the force in the radial direction should not cause any rotation, and thus no torque.

The formula for the cross product of two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ is given by

$$\begin{aligned}\overrightarrow{v} \times \overrightarrow{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \\ &= \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle.\end{aligned}$$

The cross product gives the area of the parallelogram spanned by the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$. By cutting this in half, we get to the area of the triangle, and using the law of sines, we can find that the angle between these vectors can be computed by

$$\sin\theta = \frac{||\overrightarrow{v} \times \overrightarrow{w}||}{||\overrightarrow{v}||||\overrightarrow{w}||}$$

with θ between 0 and π.

4 Planes

Planes in $\mathbb{R}^3$ can be thought of as an appropriate generalization of lines in $\mathbb{R}^2$, since lines in $\mathbb{R}^2$ are represented by equations of the form

$$ax + by = d$$

and planes in $\mathbb{R}^3$ can be represented by

$$ax + by + cz = d.$$

Geometrically, a plane consists of all points P such that the vector from a fixed base point P_0 to P is perpendicular to a fixed normal vector $\overrightarrow{n}$. Thus, we need two things to determine a plane: a base point P_0 and a normal vector $\overrightarrow{n}$. This gives the three different forms for an equation of a plane:

$$\begin{aligned}\overrightarrow{n} \cdot \langle x, y, z \rangle &= d \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ ax + by + cz &= d\end{aligned}$$

where $d = ax_0 + by_0 + cz_0$ with $\overrightarrow{n} = \langle a, b, c \rangle$ and $P_0 = (x_0, y_0, z_0)$.