SELECTED TOPICS IN REAL AND COMPLEX ANALYSIS

ABSTRACT. The purpose of this notes it to discuss selected topics in real and complex analysis and emphasize some important ideas in these fields.

1. TOPOLOGICAL PRELIMINARIES

This section is essentially based on [2]. The concepts of limit, convergence, and continuity are central to all of analysis, and it is useful to have a general framework for studying them. One such framework is that of metric spaces. However, metric spaces are not sufficiently general to describe even some very classical modes of convergence, for example, pointwise convergence of functions on $\mathbb{R}$. A more flexible theory can be built by taking the open sets as the primitive data, and this will be explored here.

1.1. **Topological spaces.** Let X be a nonempty set and I be a nonempty index set. A topology on X is a family $\mathcal{T} = \{U_\alpha \subseteq X : \alpha \in I\}$ of subsets of X that contains $\emptyset$ and X, and is closed under arbitrary unions and finite intersections. More precisely, we have:

1. $\emptyset, X \in \mathcal{T}$;
2. for every $A \subseteq I$ and $(U_\alpha : \alpha \in A) \subseteq \mathcal{T}$ we have $\bigcup_{\alpha\in A} U_\alpha \in \mathcal{T}$;
3. for any finite $A \subseteq I$ and $(U_\alpha : \alpha \in A) \subseteq \mathcal{T}$ we have $\bigcap_{\alpha\in A} U_\alpha \in \mathcal{T}$.

Example 1.1. Let us examine a few examples.

1. If X is any nonempty set, $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are topologies on X. They are called the discrete topology and the trivial topology, respectively.
2. If X is an infinite set, $\{U \subseteq X : U = \emptyset \ \text{ or } \ U^c \ \text{ is finite}\}$ is a topology on X, called the cofinite topology.
3. Recall that a metric on a set X is a function $\rho : X \times X \to [0, \infty)$ such that

   3.1. $\rho(x, y) = 0$ iff $x = y$;
   3.2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
   3.3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

   Intuitively, $\rho(x, y)$ is to be interpreted as the distance from x to y. A set equipped with a metric is called a metric space. The open ball of radius $r > 0$ centered at $x \in X$ is
   $$B(x, r) = \{y \in X : \rho(x, y) < r\}.$$
   If (X, ρ) is a metric space with a metric ρ, then the metric topology $\mathcal{T}$ is defined as the set of all $U \subseteq X$ such that U is the union of a family (empty or otherwise) of open balls in X.
4. If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$, then $\mathcal{T}(Y) = \{U \cap Y : U \in \mathcal{T}\}$ is a topology on Y, called the relative topology induced by $\mathcal{T}(Y)$.

We now present the basic terminology concerning topological spaces. $(X, \mathcal{T})$ will always denote a topological space. The members of $\mathcal{T}$ are called open sets, and their complements are called closed sets. We observe that the family of closed sets is closed under arbitrary intersections and finite

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unions. The interior of $A \subseteq X$ is the union of all open sets contained in A, and is denoted by $\mathrm{int}(A)$. The closure of $A \subseteq X$ is the intersection of all closed sets containing A, and is denoted by $\mathrm{cl}(A)$. Obviously $\mathrm{int}(A)$ is the largest open set contained in A and $\mathrm{cl}(A)$ is the smallest closed set containing A.

Exercise 1.2. Show that

$$[\mathrm{int}(A)]^c = \mathrm{cl}(A^c), \qquad \text{and} \qquad [\mathrm{cl}(A)]^c = \mathrm{int}(A^c).$$

The difference $\mathrm{cl}(A) \setminus \mathrm{int}(A) = \mathrm{cl}(\mathrm{A}) \cap \mathrm{cl}(A^c)$ is called the boundary of A and is denoted by ∂A. If $\mathrm{cl}(A) = X$, then we say that A is dense in X.

Exercise 1.3. Show that the set of all dyadic rational numbers in $[0, 1]$

$$\Delta = \{k2^{-n} : n \in \mathbb{N} \cup \{0\} \ \text{ and } \ 0 < k < 2^n\}$$

is dense in $[0, 1]$.

If $\mathrm{int}(\mathrm{cl}(A)) = \emptyset$, then we say that A is nowhere dense.

Exercise 1.4. Show that the ternary Cantor set

$$\mathcal{C} = \left\{ \sum_{n=0}^{\infty} a_n 3^{-n} : a_n \in \{0, 2\} \ \text{ for every } \ n \in \mathbb{N} \cup \{0\} \right\}$$

is nowhere dense.

If $x \in X$ (or $E \subseteq X$), a neighborhood of x (or E) is an open set $A \subseteq X$ such that $x \in A$ (or $E \subseteq A$). A point $x \in X$ is called an accumulation point of A if $A \cap (U \setminus \{x\}) \neq \emptyset$ for every neighborhood U of x.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$, or that $\mathcal{T}_2$ is stronger than $\mathcal{T}_1$. Clearly the trivial topology is the weakest topology on X, while the discrete topology is the strongest.

Exercise 1.5. Let (X, ρ) be a metric space and define a new metric by setting

$$\rho'(x, y) = \min\{1, \rho(x, y)\}.$$

Show that ρ and ρ' induce the same topology on X.

If $\mathcal{E} \subseteq \mathcal{P}(X)$, there is a unique weakest topology $\mathcal{T}(\mathcal{E})$ on X that contains $\mathcal{E}$, namely the intersection of all topologies on X containing $\mathcal{E}$. It is called the topology generated by $\mathcal{E}$, and $\mathcal{E}$ is sometimes called a subbase for $\mathcal{T}(\mathcal{E})$.

If $\mathcal{T}$ is a topology on X, a neighborhood base for $\mathcal{T}$ at $x \in X$ is a family $\mathcal{N} \subseteq \mathcal{T}$ such that

1. $x \in V$ for all $V \in \mathcal{N}$;
2. if $U \in \mathcal{T}$ and $x \in U$, there exists $V \in \mathcal{N}$ such that $x \in V$ and $V \subseteq U$.

A base for $\mathcal{T}$ is a family $\mathcal{B} \subseteq \mathcal{T}$ that contains a neighborhood base for $\mathcal{T}$ at each $x \in X$. For example, if X is a metric space, the collection of open balls centered at x is a neighborhood base for the metric topology at x, and the collection of all open balls in X is a base.

Exercise 1.6. Show that if $\mathcal{T}$ is a topology on X and $\mathcal{E} \subseteq \mathcal{T}$, then $\mathcal{E}$ is a base for $\mathcal{T}$ iff every nonempty $U \in \mathcal{T}$ is a union of members of $\mathcal{E}$.

Exercise 1.7. Show that if $\mathcal{E} \subseteq \mathcal{P}(X)$, the topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ consists of $\emptyset, X$ and all unions of finite intersections of members of $\mathcal{E}$.

Example 1.8 (Furstenberg, see [1])**.** For $a, b \in \mathbb{Z}$ with $b > 0$, we set

$$\mathbb{N}_{a,b} = \{a + bn \colon n \in \mathbb{Z}\}.$$

In other words the set $\mathbb{N}_{a,b}$ is a two-sided infinite arithmetic progression. Using the sets $\mathbb{N}_{a,b}$ we define the following curious topology on the set of integers $\mathbb{Z}$ by setting

$$\mathcal{T} = \big\{U \subseteq \mathbb{Z} \colon U = \emptyset \ \text{ or for every } a \in U \text{ there is } b \in \mathbb{N} \text{ such that } \ \mathbb{N}_{a,b} \subseteq U\big\}.$$

Clearly, the union of open sets is open again. If $U_1, U_2 \in \mathcal{T}$, and $a \in U_1 \cap U_2$ with $\mathbb{N}_{a,b_1} \subseteq U_1$ and $\mathbb{N}_{a,b_2} \subseteq U_2$ then $a \in \mathbb{N}_{a,b_1 b_2} \subseteq U_1 \cap U_2$. Finally, we see that $\mathbb{Z} = \mathbb{N}_{0,1}$. Hence, $\mathcal{T}$ is indeed a topology on $\mathbb{Z}$. Two remarks are in order:

(a) Any nonempty open set is infinite.
(b) Any set $\mathbb{N}_{a,b}$ is closed as well.

Indeed, the first fact follows from the definition. For the second we observe

$$\mathbb{N}_{a,b} = \mathbb{Z} \setminus \bigcup_{j=1}^{b-1} \mathbb{N}_{a+j,b},$$

which proves that $\mathbb{N}_{a,b}$ is the complement of an open set and hence closed.

Using this topology we prove that the set of the prime numbers $\mathbb{P}$ is infinite. Namely, we know that every $n \in \mathbb{Z} \setminus \{-1, 1\}$ has a prime divisor p, and hence is contained in $\mathbb{N}_{0,p}$, we conclude

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in \mathbb{P}} \mathbb{N}_{0,p}.$$

Now if $\mathbb{P}$ were finite, then $\bigcup_{p \in \mathbb{P}} \mathbb{N}_{0,p}$ would be a finite union of closed sets by (b), and hence closed. Consequently, $\{-1, 1\}$ would be an open set, which contradicts (a).

The concept of topological space is general enough to include a great profusion of interesting examples, but by the same reason too general to yield many interesting theorems. To build a reasonable theory one must usually restrict the class of spaces under consideration.

A topological space $(X, \mathcal{T})$ satisfies the first axiom of countability, or is first countable, if there is a countable neighborhood base for $\mathcal{T}$ at every point of X. The space $(X, \mathcal{T})$ satisfies the second axiom of countability, or is second countable, if $\mathcal{T}$ has a countable base. Also, $(X, \mathcal{T})$ is separable if X has a countable dense subset. Every metric space is first countable (the balls of rational radius about x are a neighborhood base at x), and a metric space is second countable iff it is separable.

Exercise 1.9. Every second countable space is separable.

A sequence $(x_n : n \in \mathbb{N})$ in a topological space X converges to $x \in X$, i.e. $x_n \xrightarrow[n\to\infty]{} x$ if for every neighborhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$. First countable spaces have the pleasant property that such things as closure and continuity can be characterized in terms of sequential convergence, which may not the case in more general spaces.

Exercise 1.10. If X is first countable and $A \subseteq X$, then $x \in \mathrm{cl}(A)$ iff there is a sequence $(x_n : n \in \mathbb{N})$ in A that converges to x.

In metric spaces convergence may be expressed in terms of its metric. Let (X, ρ) be a metric space, we say that a sequence $(x_n : n \in \mathbb{N}) \subseteq X$ converges to $x \in X$, i.e. $x_n \xrightarrow[n\to\infty]{} x$ or $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} \rho(x_n, x) = 0$.

Exercise 1.11. If X is a metric space, $E \subseteq X$, and $x \in X$, the following are equivalent:

(a) $x \in \mathrm{cl}(E)$;

(b) $B(x,r) \cap E \neq \emptyset$ for all $r > 0$;
(c) there is a sequence $(x_n : n \in \mathbb{N})$ in E that converges to x.

A sequence $(x_n : n \in \mathbb{N})$ in a metric space (X, ρ) is called Cauchy if $\rho(x_m, x_n) \xrightarrow[m,n\to\infty]{} 0$. A subset E of X is called complete if every Cauchy sequence in E converges and its limit is in E. For example, $\mathbb{R}^d$ (with the Euclidean metric) is complete, whereas $\mathbb{Q}^d$ is not.

Exercise 1.12. A closed subset of a complete metric space is complete, and a complete subset of an arbitrary metric space is closed.

Two metrics ρ_1 and ρ_2 on a set X are equivalent if and only if for every sequence $(x_n : n \in \mathbb{N})$ in X and every $x \in X$, we have

$$\lim_{n\to\infty} \rho_1(x_n, x) = 0 \quad \iff \quad \lim_{n\to\infty} \rho_2(x_n, x) = 0.$$

It is easily verified that equivalent metrics define the same open and closed sets.

Exercise 1.13. Show that two metrics ρ_1 and ρ_2 on a set X are equivalent iff they induce the same topology on X.

We define the diameter of $E \subseteq X$ to be

$$\operatorname{diam}(E) = \sup\{\rho(x,y) \colon x, y \in E\},$$

and we say that E is called bounded if $\operatorname{diam}(E) < \infty$.

Exercise 1.14. Show that $\operatorname{diam}(E) = \operatorname{diam}(\operatorname{cl}(E))$.

We have a nice characterization of complete metric spaces:

Theorem 1.15 (Cantor's intersection theorem)**.** *A metric space (X, ρ) is complete if and only if for every decreasing sequence $F_1 \supseteq F_2 \supseteq \ldots$ of nonempty closed subsets of X with $\operatorname{diam}(F_n) \xrightarrow[n\to\infty]{} 0$, the intersection $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$ for some $x_0 \in X$.*

Proof. Assume that (X, ρ) is complete. Let $(F_n : n \in \mathbb{N})$ be a decreasing sequence of nonempty closed sets with diameter converging to 0. Choose $x_n \in F_n$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\operatorname{diam}(F_n) < \varepsilon$ for every $n \geq N$. Note that for $n \geq m \geq N$ we have $x_n \in F_n \subseteq F_m$ so $\rho(x_m, x_n) \leq \operatorname{diam}(F_m) < \varepsilon$. This ensures that $(x_n : n \in \mathbb{N})$ is a Cauchy sequence and consequently convergent to some $x_0 \in X$. It is easily seen that $x_0 \in \bigcap_{n\in\mathbb{N}} F_n$. Suppose that there is $y \neq x_0$ such that $y \in \bigcap_{n\in\mathbb{N}} F_n$ then we have $0 < \rho(x_0, y) \leq \operatorname{diam}(F_n) \xrightarrow[n\to\infty]{} 0$, which gives a contradiction. Hence $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$. To prove the converse implication assume that $(x_n : n \in \mathbb{N})$ is a Cauchy sequence and let $F_n = \operatorname{cl}(\{x_m \colon m \geq n\})$. We immediately see that $F_1 \supseteq F_2 \supseteq \ldots$ and $\operatorname{diam}(F_n) \xrightarrow[n\to\infty]{} 0$. Thus $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$ for some $x_0 \in X$. We finally obtain that $\lim_{n\to\infty} x_n = x_0$. $\square$

1.2. **Continuous maps.** Topological spaces are the natural setting for the concept of continuity, which can be described in either global or local terms as follows. Let X and Y be topological spaces and f a map from X to Y. Then f is called continuous if $f^{-1}[U]$ is open in X for every open $V \subseteq Y$. If $x \in X$, f is called continuous at x if for every neighborhood V of $f(x)$ there is a neighborhood U of x such that $f[U] \subseteq V$, or equivalently, if $f^{-1}[V]$ is a neighborhood of x for every neighborhood V of $f(x)$. We shall denote the set of continuous maps from X to Y by $C(X, Y)$.

Exercise 1.16. The map $f : X \to Y$ is continuous iff f is continuous at every $x \in X$.

Exercise 1.17. If the topology on Y is generated by a family of sets $\mathcal{E}$, then $f : X \to Y$ is continuous iff $f^{-1}[V]$ is open in X for every $V \in \mathcal{E}$.

If (X_1, ρ_1) and (X_2, ρ_2) are metric spaces, a map $f : X_1 \to X_2$ is called continuous at $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho_2(f(y), f(x)) < \varepsilon$ whenever $\rho_1(x, y) < \delta$ in other words, such that $f^{-1}[B(f(x), \varepsilon)] \supseteq B(x, \delta)$. The map f is called continuous if it is continuous at each $x \in X$ and uniformly continuous if, in addition, the δ in the definition of continuity can be chosen independent of x.

If $f : X \to Y$ is bijective and f and f^{-1} are both continuous, f is called a homeomorphism, and X and Y are said to be homeomorphic. In this case the set mapping f^{-1} is a bijection from the open sets in Y to the open sets in X, so X and Y may be considered identical as far as their topological properties go. If $f : X \to Y$ is injective but not surjective, and $f : X \to f[X]$ is a homeomorphism when $f[X] \subseteq Y$ is given the relative topology, f is called an embedding.

We shall be particularly interested in real-valued and complex-valued functions on topological spaces (in our case it will be mainly metric spaces). If X is any set, we denote by $B(X, \mathbb{R})$ (resp. $B(X, \mathbb{C})$) the space of all bounded real-valued (resp. complex-valued) functions on X. If X is a topological space, we also have the spaces $C(X, \mathbb{R})$ and $C(X, \mathbb{C})$ of continuous functions on X, and we define $BC(X, F) = B(X, F) \cap C(X, F)$ ($F = \mathbb{R}$ or $F = \mathbb{C}$). If $f \in B(X, \mathbb{C})$, we define the uniform norm of f to be

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

The function $\rho(f, g) = \|f - g\|_\infty$ is easily seen to be a metric on $B(X, \mathbb{C})$, and convergence with respect to this metric is simply uniform convergence on X. $B(X, \mathbb{C})$ is obviously complete in the uniform metric.

Exercise 1.18. If X is a topological space, $BC(X, \mathbb{C})$ is a closed subspace of $B(X, \mathbb{C})$ in the uniform metric; in particular, $BC(X, \mathbb{C})$ is complete.

1.3. **Separation axioms.** We now discuss the separation axioms. These are properties of a topological space, labeled $T_0, T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4$ that guarantee the existence of open sets that separate points or closed sets from each other. If X has the property T_j, we say that X is a T_j space or that the topology on X is T_j.

T_0 : If $x \neq y$, there is an open set containing x but not y or an open set containing y but not x.

T_1 : If $x \neq y$, there is an open set containing y but not x.

T_2 : If $x \neq y$, there are disjoint open sets U, V with $x \in U$ and $y \in V$. A T_2 space is also called a Hausdorff space.

T_3 : X is a T_1 space, and for any closed set $A \subseteq X$ and any $x \in A^c$ there are disjoint open sets U, V with $x \in U$ and $A \subseteq V$. A T_3 space is also called a regular space.

$T_{3\frac{1}{2}}$: X is T_1 and for each closed $A \subseteq X$ and each $x \in A^c$ there exists $f \in C(X, [0, 1])$ such that $f(x) = 1$ and $f = 0$ on A. A $T_{3\frac{1}{2}}$ space is also called a completely regular space or a Tychonoff space.

T_4 : X is a T_1 space, and for any disjoint closed sets A, B in X there are disjoint open sets U, V with $A \subseteq U$ and $B \subseteq V$. A T_4 space is also called a normal space.

Exercise 1.19. X is a T_1 space iff $\{x\}$ is closed for every $x \in X$.

Exercise 1.20. Show that $T_4 \Longrightarrow T_{3\frac{1}{2}} \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0$.

The vast majority of topological space that arise in practice are Hausdorff, or become Hausdorff after simple modifications. For instance, the space of integrable functions with p-th power for

$p \in [1, \infty]$ on a measure space $(X, \mathrm{d}\mu)$ becomes a Hausdorff space with the $L^p(X)$ metric $\rho(f, g) = \big(\int_X |f - g|^p \mathrm{d}\mu\big)^{1/p}$ when we identify two functions that are equal almost everywhere on X.

For a given topological space X it may happen that $C(X, \mathbb{C})$ consists only of constant functions. This is obviously the case if X is endowed with the trivial topology, but it can happen even when X is regular. However, normal spaces have plenty of continuous functions due to Urysohn's lemma.

Theorem 1.21 (Urysohn's lemma)**.** *Let X be a normal space. Suppose A, B are two nonempty, disjoint closed subsets of X. Then there is a continuous function $f : X \to [0, 1]$ such that*

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B. \end{cases}$$

Proof. Let Δ be the set of all dyadic rational numbers in $[0, 1]$ as in Exercise 1.3. We first prove the following claim:

Claim 1.22. *There is a family $\{U_r \subseteq X : r \in \Delta\}$ of open sets such that $A \subseteq U_r \subseteq B^c$ for all $r \in \Delta$ and $\mathrm{cl}(U_r) \subseteq U_s$ for $r < s$.*

Proof of the Claim. By normality, there exist disjoint open sets V, W such that $A \subseteq V$, $B \subseteq W$. Let $U_{1/2} = V$. Then since W^c is closed, we have

$$A \subseteq U_{1/2} \subseteq \mathrm{cl}(U_{1/2}) \subseteq W^c \subseteq B^c.$$

We now select U_r for $r = k2^{-n}$ by induction on n. Suppose that we have chosen U_r for $r = k2^{-n}$ when $0 < k < 2^n$ and $n \leq N - 1$. To find U_r for $r = (2j + 1)2^{-N}$ $(0 \leq j < 2^{N-1})$, observe that $\mathrm{cl}(U_{j2^{1-N}})$ and $(U_{(j+1)2^{1-N}})^c$ are disjoint closed sets (where we set $\mathrm{cl}(U_0) = A$ and $U_1^c = B$), so as above we can choose an open U_r with

$$A \subseteq \mathrm{cl}(U_{j2^{1-N}}) \subseteq U_r \subseteq \mathrm{cl}(U_r) \subseteq U_{(j+1)2^{1-N}} \subseteq B^c.$$

This completes the proof of the claim. $\square$

Let U_r be as in the claim above for $r \in \Delta$, and set $U_1 = X$. For $x \in X$, define

$$f(x) = \inf\{r \in \Delta \colon x \in U_r\}.$$

Since $A \subseteq U_r \subseteq B^c$ for $r \in \Delta$ we clearly have

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B. \end{cases}$$

and $0 \leq f(x) \leq 1$ for all $x \in X$. It remains to show that f is continuous. To this end, observe that $f(x) < \alpha$ iff $x \in U_r$ for some $r < \alpha$ iff $x \in \bigcup_{r<\alpha} U_r$, so $f^{-1}[(-\infty, \alpha)] = \bigcup_{r<\alpha} U_r$ is open. Also $f(x) > \alpha$ iff $x \notin U_r$ for some $r > \alpha$ iff $x \notin \mathrm{cl}(U_s)$ for some $s > \alpha$ (since $U_r \subseteq \mathrm{cl}(U_s)$ for $r < s$) iff $x \in \bigcup_{s>\alpha} \mathrm{cl}(U_s)^c$, so $f^{-1}[(\alpha, \infty)] = \bigcup_{s>\alpha} \mathrm{cl}(U_s)^c$ is open. Hence f is continuous, since the open half-lines generate the topology on $\mathbb{R}$. The proof of Urysohn's lemma is completed. $\square$

Remark 1.23. If (X, ρ) is a metric space then the Urysohn lemma has a very simple proof. Namely, recall that the distance from a point to a set is defined by setting

$$\rho(x, E) = \inf\{\rho(x, y) \colon x, y \in E\}.$$

It suffices to take

$$f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)}$$

and we are done. This also shows that every metric space is normal.

Exercise 1.24. Prove that $\rho(x, E) = 0$ iff $x \in \mathrm{cl}(E)$.

Exercise 1.25. Prove that for every $x, y \in X$ we have

$$|\rho(x, E) - \rho(y, E)| \leq \rho(x, y).$$

Exercise 1.26. Let X be a normal space. If A is a closed subset of X and $f \in C(A, [a, b])$, there exists $F \in C(X, [a, b])$ such that $F_{|A} = f$.

1.4. **Product topology.** If X is any set and A an index set, and $\mathcal{F}_A = \{f_\alpha : X \to Y_\alpha : \alpha \in A\}$ is a family of maps from X into some topological spaces Y_α, there is a unique weakest topology $\mathcal{T}$ on X that makes all the f_α continuous; it is called the weak topology generated by $\mathcal{F}_A$. Namely, $\mathcal{T}$ is the topology generated by sets of the form $f_\alpha^{-1}[U_\alpha]$, where $\alpha \in A$ and U_α is open in Y_α.

The most important example of this construction is the Cartesian product of topological spaces. If $\{X_\alpha : \alpha \in A\}$ is any family of topological spaces, the product topology on $X = \prod_{\alpha\in A} X_\alpha$ is the weak topology generated by the coordinate maps $\pi_\alpha : X \to X_\alpha$. When we consider a Cartesian product of topological spaces, we always endow it with the product topology unless we specify otherwise. A base for the product topology is given by the sets of the form $\bigcap_{j=1}^{n} \pi_{\alpha_j}^{-1}[U_{\alpha_j}]$, where $n \in \mathbb{N}$ and U_{α_j} is open in X_{α_j} for $1 \leq j \leq n$. These sets can also be written as $\prod_{\alpha\in A} U_\alpha$ where $U_\alpha = X_\alpha$ if $\alpha \neq \alpha_1, \ldots, \alpha_n$. Notice, in particular, that if A is infinite, a product of nonempty open sets $\prod_{\alpha\in A} U_\alpha$ is open in $\prod_{\alpha\in A} X_\alpha$ iff $U_\alpha = X_\alpha$ for all but finitely many α.

Example 1.27. Let $(X_1, \rho_1), (X_2, \rho_2), \ldots$ be metric spaces and $X = \prod_{n=1}^{\infty} X_n$ be its product. For any $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in X$ define

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{\rho(x_n, y_n), 1\}.$$

It is not difficult to show that ρ is a metric on X, which defines the product topology on X. This metric will be called the product metric on X.

Exercise 1.28. If X_α is Hausdorff for each $a \in A$, then $X = \prod_{\alpha\in A} X_\alpha$ is Hausdorff.

Exercise 1.29. If $(X_\alpha : \alpha \in A)$ and Y are topological spaces and $X = \prod_{\alpha\in A} X_\alpha$ then $f : Y \to X$ is continuous iff $\pi_\alpha \circ f$ is continuous for each $\alpha \in A$.

If the spaces X_α are all equal to some fixed space X, the product $X = \prod_{\alpha\in A} X_\alpha$ is just the set X^A of mappings from A to X, and the product topology is just the topology of pointwise convergence. More precisely:

Exercise 1.30. If X is a topological space, A is a nonempty set, and $(f_n : n \in \mathbb{N})$ is a sequence in X^A, then $f_n \xrightarrow[n\to\infty]{} f$ in the product topology iff $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ for every $x \in X$.

When is a topological space metrizable, that is, when is its topology defined by a metric?

Theorem 1.31 (The Urysohn metrization theorem)**.** *Every second countable normal space is metrizable.*

Proof. Since every subset of a metrizable space is metrizable (with the same metric), it suffices to find a metric space in which the space in question can be seen as its subset. In fact we show that any second countable normal space X can be embedded in the Hilbert cube

$$\mathbb{H} = [0, 1] \times [0, 1] \times \ldots .$$

Let $\mathcal{B} = \{B_1, B_2, \ldots\}$ be a countable basis of X. Take $x \in X$ and U be any neighborhood of x. Since $\mathcal{B}$ is a basis of X there must be $n \in \mathbb{N}$ such that $x \in B_n \subseteq U$. But X is normal so we find disjoint open sets V, W such that $\{x\} \subseteq V$ and $B_n^c \subseteq W$, since the sets $\{x\}$ and B_n^c are closed and disjoint. Thus we get $x \in V \subseteq \mathrm{cl}(V) \subseteq W^c \subseteq B_n$ and consequently we find $m \in \mathbb{N}$ such that $x \in \mathrm{cl}(B_m) \subseteq B_n \subseteq U$.

Let us consider all pairs $(m, n) \in \mathbb{N}$ such that $\mathrm{cl}(B_m) \subseteq B_n$. By Urysohn's lemma (see Theorem 2.26) we may find continuous functions $f_{m,n} : X \to [0, 1]$ such that

$$f_{m,n}(x) = \begin{cases} 1 & \text{if } x \in \mathrm{cl}(B_m), \\ 0 & \text{if } x \in B_n^c. \end{cases}$$

Let $(g_n : n \in \mathbb{N})$ be an enumeration of the set $\{f_{m,n} \colon (m, n) \in \mathbb{N}^2\}$. We have proved that the sequence $(g_n : n \in \mathbb{N})$ has the property that for every $x \in X$ and any open U containing x, there is a continuous function $g_n : X \to [0, 1]$ such that $g_n(x) = 1$ and $g_n(y) = 0$ for every $y \in U^c$. Let us define a map

$$g = (g_1, g_2, \ldots) : X \to \mathbb{H}.$$

We have to check that g embeds X in the Hilbert cube $\mathbb{H}$. It is easy to see that $g : X \to \mathbb{H}$ is continuous, since every g_n is. We also see that g is injective. Indeed, take $x, y \in X$ so that $x \neq y$ then we can find a function $g_n : X \to [0, 1]$ such that $g_n(x) = 1$ and $g_n(y) = 0$, and we are done. Finally we will show that g is an open mapping, i.e. it maps open sets to open sets, this will ensure that $g^{-1} : g[X] \to X$ is a continuous map as well. For this purpose take an open set $U \subseteq X$. We have to show that $g[U]$ is open in $g[X]$. Take $y \in g[U]$ then $y = g(x)$ for some $x \in U$. We also see that there is $n \in \mathbb{N}$ such that $g_n(x) = 1$ and $g_n(z) = 0$ for every $z \in U^c$. Let $\pi_n : \mathbb{H} \to [0, 1]$ be a projection onto n-th coordinate and define $V = \pi_n^{-1}[(0, 1]]$. We see that V is open, since π_n is continuous in the product topology and consequently $V \cap g[X]$ is open set in $g[X]$. Note that $y = g(x) \in V \cap g[X]$, indeed $\pi_n(y) = \pi_n(g(x)) = g_n(x) = 1 \in (0, 1]$ and we are done. Finally, take $w \in V \cap g[X]$ then $w = g(v)$ for some $v \in X$ and we have $\pi_n(w) \in (0, 1]$ and $\pi_n(w) = g_n(v) \in (0, 1]$ thus $v \in U$, since $g_n(z) = 0$ for every $z \in U^c$. This shows $V \cap g[X] \subseteq g[U]$ and completes the proof of the theorem. □

Exercise 1.32. Show that the ordinary ternary Cantor set is homeomorphic to the space $X = \{0, 1\}^{\mathbb{N}}$.

1.5. Metric spaces.

We now pay our attention on metric spaces.

Theorem 1.33. *Every metric space is embeddable in a completely metrizable space.*

Proof. We fix a metric space (X, ρ), we know that the space $BC(X, \mathbb{R})$ with the topology of uniform convergence is complete. We fix $a \in X$ and assign to every $x \in X$ the function

$$f_x(z) = \rho(z, x) - \rho(z, a) \quad \text{for} \quad z \in X.$$

Since $|f_x(z)| \leq \rho(x, a)$ we have $f_x \in BC(X, \mathbb{R})$ for every $x \in X$. We shall show that

$$\|f_x - f_y\|_\infty = \rho(x, y) \quad \text{for all} \quad x, y \in X.$$

For any $z \in X$ we have

$$f_x(z) - f_y(z) = \rho(z, x) - \rho(z, a) - \rho(z, y) + \rho(z, a) \leq \rho(x, y).$$

Thus we deduce $\|f_x - f_y\|_\infty \leq \rho(x, y)$. Note also that

$$f_x(y) - f_y(y) = \rho(y, x) - \rho(y, a) - \rho(y, y) + \rho(y, a) = \rho(y, x).$$

Thus $\|f_x - f_y\|_\infty \geq \rho(x, y)$ and the proof is completed. □

Theorem 1.34 (Baire's category theorem)**.** *Assume that (X,ρ) is a complete metric space. If $\{U_n : n \in \mathbb{N}\}$ is a countable family of open dense sets in X, then $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.*

Proof. Let V be a nonempty open set in X. We show that $V \cap \bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$. Since U_1 is dense, $U_1 \cap V \neq \emptyset$. Choose an open ball B_1 of diameter < 1 such that $\mathrm{cl}(B_1) \subseteq U_1 \cap V$. Since U_1 is open and dense, by the same argument we get an open ball B_2 of diameter $< 1/2$ such that $\mathrm{cl}(B_2) \subseteq U_1 \cap B_1$. Proceeding similarly, we define a sequence $(B_n : n \in \mathbb{N})$ of open balls in X such that for each $n \in \mathbb{N}$, we have

(a) $\mathrm{diam}(B_n) < 2^{-n-1}$;
(b) $\mathrm{cl}(B_1) \subseteq U_1 \cap V$;
(c) $\mathrm{cl}(B_{n+1}) \subseteq U_n \cap B_n$.

Since (X,ρ) is a complete metric space, by Cantor's theorem $\bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} \mathrm{cl}(B_n) = \{x\}$ for some $x \in X$. Clearly, $x \in V \cap \bigcap_{n\in\mathbb{N}} U_n$. □

Baire's theorem is a very important result in mathematics, which often used in analysis to prove results that have existential nature. The name of this theorem comes from Baire's terminology for sets: If X is a topological space, a set $E \subseteq X$ is of the first category, according to Baire, if E is a countable union of nowhere dense sets; otherwise E is of the second category.

Corollary 1.35. *Every completely metrizable space is of the second category in itself.*

Proof. Let X be a completely metrizable space. Suppose X is of the first category in itself. Choose a sequence $(F_n : n \in \mathbb{N})$ of closed and nowhere dense sets such that $X = \bigcup_{n\in\mathbb{N}} F_n$. Then the sets $U_n = F_n^c$ are dense and open, and $\bigcap_{n\in\mathbb{N}} U_n = \emptyset$. This contradicts the Baire category theorem. □

Theorem 1.36 (Dual form of Baire's category theorem)**.** *Assume that (X,ρ) is a complete metric space. If $\{F_n : n \in \mathbb{N}\}$ is a countable family of nowhere dense sets in X, then $\bigcup_{n\in\mathbb{N}} F_n$ has empty interior.*

Proof. Since $\mathrm{int}(\mathrm{cl}\,(F_n)) = \mathrm{int}(F_n) = \emptyset$ for every $n \in \mathbb{N}$ observe

$$\mathrm{int}\Big(\bigcup_{n\in\mathbb{N}} F_n\Big) \subseteq \mathrm{int}\Big(\bigcup_{n\in\mathbb{N}} \mathrm{cl}(F_n)\Big) = \left(\mathrm{cl}\left(\bigcap_{n\in\mathbb{N}} \mathrm{cl}(F_n)^c\right)\right)^c = X^c = \emptyset,$$

by Baire's category theorem, since $\mathrm{cl}(F_n)^c$ is open and dense for every $n \in \mathbb{N}$. □

From Baire's category theorem we obtain a nontrivial result for the rational numbers.

Corollary 1.37. *The set of rationals $\mathbb{Q}$ with the Euclidean topology is not completely metrizable.*

Proof. Suppose for a contradiction that $\mathbb{Q}$ is a complete space with the Euclidean metric. Then

$$\mathbb{Q} = \{q_n : n \in \mathbb{N}\} = \bigcup_{n\in\mathbb{N}} \{q_n\},$$

where each $\{q_n\}$ is nowhere dense in $\mathbb{Q}$, so by Baire's category theorem $\mathbb{Q}$ is the set of the first category, which is impossible for complete metric spaces. □

Unfortunately, Baire's category theorem does not allow us to decide whether the set of irrational numbers is completely metrizable. However, the next theorem provides a pretty satisfactory answer.

Theorem 1.38 (Alexandroff's theorem)**.** *Every G_δ subset G of a completely metrizable space X is completely metrizable.*

We say that a subset G of a topological space X is G_δ if it is a countable intersection of open sets. We say that $F \subseteq X$ is F_σ if it is a countable union of closed sets.

Proof. Fix a complete metric ρ on X compatible with its topology. Let $G = \bigcap_{n\in\mathbb{N}} G_n$, where G_n is open set in X for every $n \in \mathbb{N}$. For every $y \in G$ let $\rho(y, F_n)$ be the distance from the closed set $F_n = X \setminus G_n$. We know that $y \mapsto \rho(y, F_n)$ are continuous. For every $n \in \mathbb{N}$ let $f_n : G \to \mathbb{R}$ be defined by

$$f_n(y) = \frac{1}{\rho(y, F_n)}.$$

Note that f_n are continuous and well defined, since $\rho(y, F_n) \neq 0$ for every $y \in G$. We also define a map $f : G \to \mathbb{R}^{\mathbb{N}}$ by setting

$$f(y) = (f_1(y), f_2(y), \ldots).$$

We immediately see that f is continuous since all f_n's are. Then it turns out that

$$\mathrm{Graph}(f) = \{(y, f(y)) \in G \times \mathbb{R}^{\mathbb{N}} \colon y \in G\}$$

the graph of f is a closed subset of $X \times \mathbb{R}^{\mathbb{N}}$. Indeed, suppose that a sequence $(y_k, f(y_k)) \in \mathrm{Graph}(f)$ converges to a point $(y, x_1, x_2, \ldots) \in X \times \mathbb{R}^{\mathbb{N}}$ as $k \to \infty$. We have to show that $(y, x_1, x_2, \ldots) \in \mathrm{Graph}(f)$. For every $n \in \mathbb{N}$ we have

$$\lim_{k\to\infty} y_k = y \quad \text{and} \quad \lim_{k\to\infty} \frac{1}{\rho(y_k, F_n)} = x_n.$$

By the continuity of $\rho(y_k, F_n)$ and the fact that the fractions $\frac{1}{\rho(y_k,F_n)}$ converge to a finite number x_k as $k \to \infty$ we deduce

$$\lim_{k\to\infty} \rho(y_k, F_n) = \rho(y, F_n) \neq 0.$$

Thus we obtain $y \notin F_n$, hence $y \in G_n$ for every $n \in \mathbb{N}$, which proves that $y \in G$. Moreover,

$$\lim_{k\to\infty} \frac{1}{\rho(y_k, F_n)} = \frac{1}{\rho(y, F_n)} = x_n$$

and consequently we obtain

$$(y, x_1, x_2, \ldots) = \left(y, \frac{1}{\rho(y, F_1)}, \frac{1}{\rho(y, F_2)}, \ldots\right) = (y, f(y)) \in \mathrm{Graph}(f).$$

So we have proved that $\mathrm{Graph}(f)$ is a closed subset of $X \times \mathbb{R}^{\mathbb{N}}$. This shows that $\mathrm{Graph}(f)$ is complete space as a closed subset of a complete space. Since $X \times \mathbb{R}^{\mathbb{N}}$ is a complete space as a countable Cartesian product of complete spaces. Finally, we conclude that G is completely metrizable, since the set $\mathrm{Graph}(f)$ is homeomorphic with its domain, which is G. □

From the above theorem we see that the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$, with the usual topologies, are completely metrizable, though the usual metrics may not be complete on them.

The converse of Alexandroff's theorem is also true, in the following form.

Exercise 1.39. If a subset E of a metric space (X, ρ) is homeomorphic to a complete metric space (Y, θ), then E is a G_δ subset of X.

We now show more consequences from Baire's category theorem.

Lemma 1.40. *Let E be a closed subset of a metric space (X, ρ). Then the following are equivalent:*

(a) E is nowhere dense;
(b) for every $x \in E$ and every $\varepsilon > 0$ there is $y \in E^c$ such that $\rho(x, y) \leq \varepsilon$.

Proof. Assume that E is nowhere dense. For every $x \in E$ we must have $B(x,\varepsilon) \cap E^c \neq \emptyset$, since otherwise $B(x,\varepsilon) \subseteq E$ and this would imply $\emptyset \neq B(x,\varepsilon) \subseteq \mathrm{int}(E) = \emptyset$. We now prove the converse implication. Since $E = \mathrm{cl}(E)$ it suffices to prove that $\mathrm{int}(E) = \emptyset$. If we had $B(x,\varepsilon) \subseteq \mathrm{int}(E)$ for some $x \in \mathrm{int}(E)$ and $\varepsilon > 0$ then $B(x,\varepsilon) \cap E^c = \emptyset$, but this is impossible in view of (b). $\square$

Proposition 1.41. *Let $C([0,1],\mathbb{R})$ be equipped with the uniform convergence topology. The set of all nowhere differentiable continuous functions is of the second category in $C([0,1],\mathbb{R})$. In particular, there exist continuous functions on $[0,1]$ which are nowhere differentiable.*

From Newton's time through the early part of the nineteenth century, most mathematicians assumed that a continuous real-valued function defined on an interval in the real line must be differentiable over most of its domain. In 1834, Bernhard Bolzano gave an example of a real-valued function continuous on an interval though differentiable nowhere on that interval, but for almost a century afterward mathematicians treated such functions as pathological. However, in 1931 Stefan Banach showed that, in a sense, the vast majority of continuous scalar-valued functions whose domain is a given interval in $\mathbb{R}$ are not differentiable anywhere.

Proof of Proposition 1.41. We must show that the set $\mathcal{D}$, of continuous functions in $[0,1]$ that are differentiable at least at one point, is of the first category. To this end, for any rational numbers $u < v$ in $[0,1]$ and any $n \in \mathbb{N}$ let

$$\mathcal{D}(u,v,n) = \{f \in C([0,1],\mathbb{R}) \colon \exists_{x^* \in [u,v]} \forall_{x \in [u,v]}\ |f(x) - f(x^*)| \leq n|x - x^*|\}.$$

To prove the claim it suffices to show that

(a) We have

$$\mathcal{D} \subseteq \bigcup_{u \in \mathbb{Q}} \bigcup_{\substack{v \in \mathbb{Q} \\ v > u}} \bigcup_{n \in \mathbb{N}} \mathcal{D}(u,v,n).$$

(b) Each $\mathcal{D}(u,v,n)$ is closed.

(c) The interior of each $\mathcal{D}(u,v,n)$ is empty.

The property (b) and (c) will guarantee that the set $\mathcal{D}(u,v,n)$ is nowhere dense and by Baire's category theorem and property (a) the desired conclusion will follow.

Proof of property (a). If $f \in \mathcal{D}$ then there exists $x_0 \in [0,1]$ such that the derivative $f'(x_0)$ exists and for $x \in [0,1]$ we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x),$$

where r is a function satisfying

$$\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0.$$

Therefore, we can always find rational numbers $0 \leq u < v \leq 1$ such that $x_0 \in [u,v]$ and $|r(x)| \leq |x - x_0|$ for every $x \in [u,v]$. Let $n \in \mathbb{N}$ be such that $n \geq |f'(x_0)| + 1$, then we have

$$|f(x) - f(x_0)| \leq |f'(x_0)||x - x_0| + |r(x)| \leq n|x - x_0|.$$

Proof of property (b). Let $(f_n : n \in \mathbb{N})$ be a sequence of functions in $\mathcal{D}(u,v,n)$ such that $\lim_{m\to\infty}\|f_m - f\|_\infty = 0$ for some $f \in C([0,1],\mathbb{R})$. We have to show that $f \in \mathcal{D}(u,v,n)$. Since $f_m \in \mathcal{D}(u,v,n)$ then there is $x_m^* \in [u,v]$ such that

$$|f_m(x) - f_m(x_m^*)| \leq n|x - x_m^*|$$

for all $x \in [u,v]$. By passing to a subsequence we can assume without loss of generality that $\lim_{m\to\infty} x_m^* = x^*$ for some $x^* \in [u,v]$. Note that for every $x \in [u,v]$ we have

$$\begin{aligned} |f(x) - f(x^*)| &\le |f(x) - f_m(x)| + |f_m(x) - f_m(x^*)| + |f_m(x^*) - f(x^*)| \\ &\le 2\|f_m - f\|_\infty + |f_m(x) - f_m(x_m^*)| + |f_m(x_m^*) - f_m(x^*)| \\ &\le 2\|f_m - f\|_\infty + n|x - x_m^*| + n|x_m^* - x^*|. \end{aligned}$$

Passing with $m \to \infty$, we obtain

$$|f(x) - f(x^*)| \le n|x - x^*|,$$

as desired. This proves that $\mathcal{D}(u,v,n)$ is closed.
Proof of property (c). We shall use (b) from Lemma 1.40 to show that $\operatorname{int}(\mathcal{D}(u,v,n)) = \emptyset$. For each $N > 0$, let $\mathcal{P}_N$ denote the set of all continuous piecewise-linear functions, each of whose line segments have slopes either $\ge N$ or $\le -N$. We will need the following:

Claim 1.42. *For every $N > 0$ the set $\mathcal{P}_N$ is dense in $C([0,1],\mathbb{R})$.*

Proof of Claim 1.42. It is not difficult to see that for every $f \in C([0,1],\mathbb{R})$ and every $\varepsilon > 0$ there exists a piecewise-linear function $g \in C([0,1],\mathbb{R})$ such that $\|f-g\|_\infty \le \varepsilon$. Indeed, f is continuous on the compact set $[0,1]$ it must be uniformly continuous. Thus we find $\delta > 0$ such that $|f(x)-f(y)| \le \frac{\varepsilon}{2}$ whenever $|x-y| < \delta$. Taking $m \in \mathbb{N}$ so that $\frac{1}{m} < \delta$ we define g as a linear function on each interval $[\frac{k}{m}, \frac{k+1}{m}]$ for any $k \in \{0,1,\ldots,m-1\}$ with $g(\frac{k}{m}) = f(\frac{k}{m})$ and $g(\frac{k+1}{m}) = f(\frac{k+1}{m})$. Then we obtain that $\|f-g\|_\infty \le \frac{\varepsilon}{2}$. We now approximate g by a function from $\mathcal{P}_N$. Indeed, g is linear on the interval $[0, \frac{1}{m}]$ and we consider two functions

$$\varphi_\varepsilon(x) = g(x) + \varepsilon/2 \quad \text{and} \quad \psi_\varepsilon(x) = g(x) - \varepsilon/2.$$

Then beginning at $g(0)$ we travel on a line segment of slope N until we intersect φ_ε. Then, we reverse direction and travel on a line segment of slope $-N$ until we intersect ψ_ε. We obtain a function $h \in \mathcal{P}_N$ so that

$$\psi_\varepsilon(x) \le h(x) \le \varphi_\varepsilon(x) \quad \text{for all} \quad x \in \left[0, \frac{1}{m}\right],$$

and consequently $|h(x) - g(x)| \le \frac{\varepsilon}{2}$ for all $x \in [0, \frac{1}{m}]$. Then we begin at $h(\frac{1}{m})$ and repeat this argument on the interval $[\frac{1}{m}, \frac{2}{m}]$. Continuing in this fashion we obtain a function $h \in \mathcal{P}_N$ such that $\|h-g\|_\infty \le \frac{\varepsilon}{2}$. Then we conclude that $\|f-h\|_\infty \le \varepsilon$ and the claim is proved. ☐

We fix $N > n$, by Claim 1.42, for every $f \in \mathcal{D}(u,v,n)$ and every $\varepsilon > 0$, we find a function $h \in \mathcal{P}_N$ such that $\|f-h\|_\infty \le \varepsilon$. We note that $\mathcal{D}(u,v,n) \cap \mathcal{P}_N = \emptyset$ if $N > n$. Invoking Lemma 1.40 we obtain that $\operatorname{int}(\mathcal{D}(u,v,n)) = \emptyset$. ☐

Remark 1.43. Finally we emphasize that if one is familiar with the theory of Fourier series it is possible to show that the following function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k} e^{2\pi i 3^k x}$$

is nowhere differentiable on the unit circle.

1.6. **Compactness.** Let E be a subset of a topological space X. If $\{U_\alpha : \alpha \in A\}$ is a family of sets such that $E \subseteq \bigcup_{\alpha \in A} U_\alpha$, then $\{U_\alpha : \alpha \in A\}$ is called a cover of E, and E is said to be covered by the U_α's. We say that a topological space X is compact if every open cover $\{U_\alpha : \alpha \in A\}$ of X — that is, a collection of open sets such that $X = \bigcup_{\alpha \in A} U_\alpha$ — has a finite subcover, which means that there is a finite subset B of A such that $X = \bigcup_{\alpha \in B} U_\alpha$.

A subset Y of a topological space X is called compact if it is compact in the relative topology; thus $Y \subseteq X$ is compact iff whenever $\{U_\alpha : \alpha \in A\}$ is a collection of open subsets of X with $Y \subseteq \bigcup_{\alpha \in A} U_\alpha$, there is a finite $B \subseteq A$ with $Y \subseteq \bigcup_{\alpha \in B} U_\alpha$.

It is easy to see that we are led to the following characterization of compactness in terms of closed sets. A family $\{F_\alpha : \alpha \in A\}$ of subsets of X is said to have the finite intersection property if $\bigcap_{\alpha \in B} F_\alpha \neq \emptyset$ for all finite $B \subseteq A$.

Exercise 1.44. Show that a topological space X is compact iff for every family $\{F_\alpha : \alpha \in A\}$ of closed sets with the finite intersection property, $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$.

We now list several basic facts about compact spaces.

Exercise 1.45. A closed subset of a compact space is compact.

Exercise 1.46. If F is a compact subset of a Hausdorff space X and $x \notin F$, there are disjoint open sets U, V such that $x \in U$ and $F \in V$.

Exercise 1.47. Every compact subset of a Hausdorff space is closed.

We remark that in a non-Hausdorff space, compact sets need not be closed (for example, every subset of a space with the trivial topology is compact), and the intersection of compact sets need not be compact. Of course, in a Hausdorff space the intersection of any family of compact sets is always compact.

Exercise 1.48. Every compact Hausdorff space is normal.

Exercise 1.49. If X is compact and $f : X \to Y$ is continuous, then $f[X]$ is compact. We also have $C(X, \mathbb{C}) = BC(X, \mathbb{C})$.

Exercise 1.50. If X is compact and Y is Hausdorff, then any continuous bijection $f : X \to Y$ is a homeomorphism.

In a metric space (X, ρ) a set $E \subseteq X$ is called totally bounded if, for every $\varepsilon > 0$, E can be covered by finitely many balls of radius ε. It is easy to see that every totally bounded set is bounded, and if E is totally bounded, so is $\mathrm{cl}(E)$. We now give a characterization of compact sets in metric spaces.

Exercise 1.51. If E is a subset of the metric space (X, ρ), the following are equivalent:

(a) E is complete and totally bounded.
(b) (The Bolzano–Weierstrass property) Every sequence in E has a subsequence that converges to a point of E.
(c) (The Heine–Borel property) If $\{U_\alpha : \alpha \in A\}$ is a cover of E by open sets, there is a finite set $B \subseteq A$ such that $\{U_\alpha : \alpha \in B\}$ covers E.

A set E that possesses the properties (a)-(c) of the previous exercise is called compact. Every compact set is closed and bounded; the converse is false in general but true in $\mathbb{R}^d$.

Exercise 1.52. Every closed and bounded subset of $\mathbb{R}^d$ is compact.

Theorem 1.53 (Alexander's subbase theorem)**.** *Let $(X, \mathcal{T})$ be a topological space and $\mathcal{E}$ be a subbase for $\mathcal{T}$. If every collection of sets from $\mathcal{E}$ that covers X has a finite subcover, then X is compact.*

Proof. The proof is by contradiction. Suppose every cover of X by sets in $\mathcal{E}$ has a finite subcover and X is not compact. Then the collection $\mathcal{F}$ of all open covers of X with no finite subcover is nonempty and partially ordered by set inclusion. With an eye towards Zorn's Lemma, take any totally ordered subset $\{E_\alpha : \alpha \in A\}$ in $\mathcal{F}$. Then we claim $E = \bigcup_{\alpha\in A} E_\alpha$ is an upper bound. To see that E contains no finite subcover, look at any finite subcollection $U_1, \dots, U_n$. Then for every $1 \le j \le n$ we have $U_j \in E_{\alpha_j}$ for some α_j. Since we have a total ordering, there is some E_{α_0} that contains all of the U_j. Thus, this finite subcollection cannot cover X.

Now Zorn's Lemma gives us a maximal element $\mathcal{M}$ of $\mathcal{F}$. Consider the set $S = \mathcal{M} \cap \mathcal{E}$. We claim that S is a cover of X. If not, we can find some $x \in X$ that is not in any of the members of S. Since $\mathcal{M}$ does cover X, there is some $U \in \mathcal{M}$ with $x \in U$. Since $\mathcal{E}$ is a subbase, there are $V_1 \dots, V_n$ in $\mathcal{E}$ with $x \in \bigcap_{j=1}^n V_j \subseteq U$. None of these V_j are in $\mathcal{M}$ because then x would be an element of some member of S. By maximality of $\mathcal{M}$, each $\mathcal{M} \cup \{V_j\}$ must contain a finite subcover of X, say $X = V_j \cup W_j$, where W_j is a finite union of sets in $\mathcal{M}$. Then

$$U \cup \bigcup_{j=1}^n W_j \supseteq \Big(\bigcap_{j=1}^n V_j\Big) \cup \Big(\bigcup_{j=1}^n W_j\Big) \supseteq \bigcap_{j=1}^n (V_j \cup W_j) \supseteq X.$$

This is impossible by construction of $\mathcal{M}$. Then S is a cover of X. Then because S is contained in $\mathcal{E}$, it would thus have a finite subcover by assumption. This is a contradiction however because S is contained in $\mathcal{M}$. Therefore, our original collection $\mathcal{F}$ must be empty so that X is compact. □

Theorem 1.54 (Tychonoff's theorem)**.** *If $(X_\alpha, \mathcal{T}_\alpha)$ is a compact topological space for each $\alpha \in A$ then $X = \prod_{\alpha\in A} X_\alpha$ endowed with the product topology is compact as well.*

Proof. We show that any open cover of X consisting solely of elements of the form $\pi_\alpha^{-1}[U]$, where $U \in \mathcal{T}_\alpha$ contains a finite subcover of X. Let $\mathcal{F}$ be such a cover, and define

$$\mathcal{U}_\alpha = \{U \in \mathcal{U}_\alpha \colon \pi_\alpha^{-1}[U] \in \mathcal{F}\}.$$

We claim that there is at least one $\alpha \in A$ such that $\mathcal{U}_\alpha$ covers X_α. If not, then for each $\alpha \in A$, there is some $x_\alpha \in X_\alpha$ such that x_α is not in the union of all the elements in $\mathcal{U}_\alpha$. Now define $f \in X$ via $f(\alpha) = x_\alpha$. Then f would not be contained in any of the members of $\mathcal{F}$, a contradiction since $\mathcal{F}$ is a cover of X. So choose α such that $\mathcal{U}_\alpha$ is a cover of X_α. By compactness, there are $U_1, \dots, U_n \in \mathcal{U}_\alpha$ such that $X_\alpha \subseteq \bigcup_{j=1}^n U_j$. Then a finite cover of X is given by

$$\{\pi_\alpha^{-1}[U_j] \colon 1 \le j \le n\}.$$

Take as a subbase for the product topology on X the collection

$$\mathcal{E} = \{\pi_\alpha^{-1}[U] \colon U \in \mathcal{T}_\alpha \text{ for each } \alpha \in A\}.$$

As we have shown any subcollection of this set that covers X has a finite subcover. Thus by Alexander's subbase theorem, X is compact and the proof is completed. □

If X is a topological space and $\mathcal{F} \subseteq C(X, \mathbb{C})$, $\mathcal{F}$ is called equicontinuous at $x \in X$ if for every $\epsilon > 0$ there is a neighborhood U of X such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and all $f \in \mathcal{F}$, and $\mathcal{F}$ is called equicontinuous if it is equicontinuous at each $x \in X$. Also, $\mathcal{F}$ is said to be pointwise bounded if $\{f(x) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$ for each $x \in X$.

Exercise 1.55 (Arzelá–Ascoli Theorem I)**.** Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $C(X, \mathbb{C})$, then $\mathcal{F}$ is totally bounded in the uniform metric, and the closure of $\mathcal{F}$ in $C(X, \mathbb{C})$ is compact.

Exercise 1.56 (Arzelá–Ascoli Theorem II)**.** Let X be a σ-compact LCH space. If $(f_n : n \in \mathbb{N})$ is an equicontinuous, pointwise bounded sequence in $C(X, \mathbb{C})$, there exist $f \in C(X, \mathbb{C})$ and a subsequence of $(f_n : n \in \mathbb{N})$ that converges to f uniformly on compact sets.

2. Density and equidistribution

2.1. **Density.** This section is based on [4] and [3]. A very important role in analysis is played by the fact that the set of rational numbers $\mathbb{Q}$ is countable and dense in $\mathbb{R}$. In other words every real number can be seen as a limit of a sequence of rational numbers. This fact can be deduced from the theorem stated below, which is a beautiful application of the pigeonhole principle.

Remark 2.1. The pigeonhole principle or Dirichlet's box principle states that if n objects are placed in r boxes, where $r < n$, then at least one of the boxes contains more than one object.

For any $x \in \mathbb{R}$ the integer and fractional part of x will be denoted respectively by

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\} \quad \text{ and } \quad \{x\} = x - \lfloor x \rfloor.$$

Theorem 2.2 (Dirichlet)**.** *Let α and Q be real numbers, $Q \geq 1$. There exist $a, q \in \mathbb{Z}$ such that*

$$1 \leq q \leq Q, \qquad (a, q) := \gcd(a, q) = 1,$$

and

$$|\alpha - a/q| < (qQ)^{-1}.$$

Proof. Let $N = \lfloor Q \rfloor$. Suppose that $\{q\alpha\} \in [0, 1/(N+1)]$ for some positive integer $q \leq N$. If $a = \lfloor q\alpha \rfloor$, then

$$0 \leq \{q\alpha\} = q\alpha - a < 1/(N+1),$$

and so

$$|\alpha - a/q| < q^{-1}(N+1)^{-1} < (qQ)^{-1} \leq q^{-2}.$$

Similarly, if $\{q\alpha\} \in [N/(N+1), 1)$ for some positive integer $q \leq N$ and if $a = \lfloor q\alpha \rfloor + 1$, then

$$N/(N+1) \leq \{q\alpha\} = q\alpha - a + 1 < 1,$$

implies that

$$|q\alpha - a| \leq 1/(N+1)$$

and so

$$|\alpha - a/q| < q^{-1}(N+1)^{-1} < (qQ)^{-1} \leq q^{-2}.$$

If $\{q\alpha\} \in [1/(N+1), N/(N+1))$ for all $1 \leq q \leq N$, then each of the N real numbers $\{q\alpha\}$ lies in one of the $N-1$ intervals

$$[j/(N+1), (j+1)/(N+1)), \qquad 1 \leq j \leq N-1.$$

By Dirichlet's box principle, there exist integers $j \in [1, N]$ and $q_1, q_2 \in [1, N]$ such that $1 \leq q_1 < q_2 \leq N$ and

$$\{q_1\alpha\}, \{q_2\alpha\} \in [j/(N+1), (j+1)/(N+1)).$$

Let $q = q_2 - q_1 \in [1, N-1]$ and $a = \lfloor q_2\alpha \rfloor - \lfloor q_1\alpha \rfloor$. Then

$$|q\alpha - a| = |(q_2\alpha - \lfloor q_2\alpha \rfloor) - (q_1\alpha - \lfloor q_1\alpha \rfloor)| = |\{q_2\alpha\} - \{q_1\alpha\}| < (N+1)^{-1} < Q^{-1}.$$

This completes the proof. ∎

Exercise 2.3. Using the previous theorem show that every real number is a limit of a sequence of rational numbers.

Exercise 2.4. Assume that $\alpha \notin \mathbb{R}$. Using Theorem 2.2 show that the sequence $(\{n\alpha\} : n \in \mathbb{N})$ is dense in $[0, 1]$.

Another beautiful application of the pigeonhole principle contained in a paper of Erdös and Szekeres on Ramsey problems is the following:

Exercise 2.5. In any sequence $a_1, a_2, \ldots, a_{mn+1}$ of $mn + 1$ distinct real numbers, there exists an increasing or decreasing subsequence of length $m + 1$.

We now illustrate that the density of rational numbers is very useful.

Proposition 2.6. *For every positive numbers $a_1, \ldots, a_n$ and positive weights $q_1, \ldots, q_n$ satisfying*

$$q_1 + \ldots + q_n = 1,$$

we have

$$a_1^{q_1} \cdot \ldots \cdot a_n^{q_n} \le q_1 a_1 + \ldots + q_n a_n.$$

Proof. We first assume that all weights $q_1, \ldots, q_n$ are positive rational numbers. We can assume that

$$q_i = \frac{k_i}{m}, \quad \text{for} \quad i \in \{1, \ldots, n\},$$

where k_i are integers such that $k_1 + \ldots + k_n = m$. Then invoking the inequality between arithmetic and geometric means we obtain

$$\begin{aligned} \sum_{i=1}^{n} q_i a_i &= k_1 \frac{a_1}{m} + \ldots + k_n \frac{a_n}{m} \\ &\ge m \left(\left(\frac{a_1}{m} \right)^{k_1} \cdot \ldots \cdot \left(\frac{a_n}{m} \right)^{k_n} \right)^{1/m} \\ &\ge a_1^{q_1} \cdot \ldots \cdot a_n^{q_n}. \end{aligned}$$

If now all weights $q_1, \ldots, q_n$ are arbitrary positive real numbers. We choose for every $i \in \{1, \ldots, n\}$ a sequence of positive rational numbers $(q_{i,l} : l \in \mathbb{N})$ such that

$$\lim_{l \to \infty} q_{i,l} = q_i \quad \text{and} \quad \sum_{i=1} q_{i,l} = 1.$$

Then we have

$$a_1^{q_{1,l}} \cdot \ldots \cdot a_n^{q_{n,l}} \le q_{1,l} a_1 + \ldots + q_{n,l} a_n,$$

and taking $l \to \infty$ we obtain the desired claim. $\square$

Exercise 2.7. Show that equality holds in the previous proposition iff $a_1 = \ldots = a_n$.

Exercise 2.8. Using mathematical induction show that if $a_1, a_2, \ldots, a_n$ are all positive numbers such that $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 1$, then

$$a_1 + a_2 + \ldots + a_n \ge n.$$

Using this show that for every positive numbers $x_1, x_2, \ldots, x_n$ we have

$$(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{1/n} \le \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

Remark 2.9. Proposition 2.6 can be used to prove Hölder's inequality which asserts that for any measure space $(X, \mathcal{B}(X), \mu)$ and every $1 \leq p, q \leq \infty$ if $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(X)$ and $g \in L^q(X)$ then we have

$$\|fg\|_{L^1(X)} \leq \|f\|_{L^p(X)} \|g\|_{L^q(X)}. \tag{2.10}$$

However, we show a different proof, which uses the best constant trick. We assume that $\|f\|_{L^p(X)} = \|g\|_{L^q(X)} = 1$ and observe

$$\begin{aligned}
\|fg\|_{L^1(X)} &= \int_X |fg| \mathrm{d}\mu \\
&= \int_X (|f|^p)^{1/p} (|g|^q)^{1/q} \mathrm{d}\mu \\
&= \int_X \max\{|f|^p, |g|^q\} \mathrm{d}\mu \\
&\leq \int_X |f|^p \mathrm{d}\mu + \int_X |g|^q \mathrm{d}\mu \\
&= 2\|f\|_{L^p(X)} \|g\|_{L^q(X)}.
\end{aligned}$$

Thus we have proved that

$$\|fg\|_{L^1(X)} \leq 2\|f\|_{L^p(X)} \|g\|_{L^q(X)}. \tag{2.11}$$

We now show how to improve this inequality. Let $C > 0$ be the smallest constant such that for all measure spaces $(X, \mathcal{B}(X), \mu)$ the following inequality holds

$$\|fg\|_{L^1(X)} \leq C\|f\|_{L^p(X)} \|g\|_{L^q(X)}. \tag{2.12}$$

By (2.11) we know that $C \leq 2$. We shall show that $C \leq 1$. Indeed, let $F(x, y) = f(x)f(y)$ and $G(x, y) = g(x)g(y)$ then by (2.12) we get

$$\begin{aligned}
\|fg\|_{L^1(X)}^2 &= \|FG\|_{L^1(X \times X)} \\
&\leq C\|F\|_{L^p(X \times X)} \|G\|_{L^q(X \times X)} \\
&= C\|f\|_{L^p(X)}^2 \|g\|_{L^q(X)}^2.
\end{aligned}$$

Thus we have proved inequality (2.12) with a new constant

$$\|fg\|_{L^1(X)} \leq C^{1/2} \|f\|_{L^p(X)} \|g\|_{L^q(X)}. \tag{2.13}$$

Therefore, by the definition of $C > 0$ we must have $C \leq C^{1/2}$, dividing both sides by $C^{1/2}$ (we can do this since $C \leq 2$) we finally obtain that $C \leq 1$ as claimed.

Exercise 2.14. Using Hölder's inequality show that for every $1 \leq p \leq \infty$ and $f, g \in L^p(X)$ we have

$$\|f + g\|_{L^1(X)} \leq \|f\|_{L^p(X)} + \|g\|_{L^q(X)}.$$

Example 2.15 (Gelfand)**.** We will consider the sequence of the first digits of powers of 2. Namely, for $m \in \mathbb{N}$ let

$$d_m = \text{first digit of } 2^m.$$

For instance we have $d_1 = 2$, $d_2 = 4$, $d_3 = 8$, $d_4 = 1$, $d_5 = 3, \ldots$. Here is a list of the first 20 powers of 2:

$$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192,$$
$$16384, 32768, 65536, 131072, 262144, 524288, 1048576.$$

The sequence of first digits of the first 40 powers of 2 is:

$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1,$$
$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1,$$
$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1,$$
$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1.$$

Do we ever see a 7, a 9? Gelfand's question asks: how often do we see a power of 2 that starts with a 7, and with what frequency? We show here that there are infinitely many $m \in \mathbb{N}$ such that 2^m starts with a 7. Surprisingly, they have a well-defined frequency. The existence of this frequency will follow from the uniform distribution of multiples of an irrational number modulo 1. We obtain this fact later.

The crucial observation is that the first digit of 2^m is equal to k if and only if there is a nonnegative integer s such that

$$k10^s \leq 2^m < (k+1)10^s.$$

Taking logarithms with base 10 we obtain

$$s + \log_{10} k \leq m \log_{10} 2 < s + \log_{10}(k+1),$$

but since $0 \leq \log_{10} k$ and $\log_{10}(k+1) \leq 1$, taking fractional parts we obtain that

$$s = \lfloor m \log_{10} 2 \rfloor$$

and that

$$\log_{10} k \leq m \log_{10} 2 - \lfloor m \log_{10} 2 \rfloor < \log_{10}(k+1). \tag{2.16}$$

Since the number $\log_{10} 2$ is irrational, it follows that the sequence $(\{m \log_{10} 2\} : m \in \mathbb{N})$ is dense in $[0, 1]$. Therefore, there are infinitely many $m \in \mathbb{N}$ such that (2.16) holds. This gives an affirmative answer to the first part of Gelfand's question. We shall handle the second part later in this section.

2.2. **Weierstrass's theorem.** In this section we present two variants of Weierstrass's theorem concerning a uniform approximation of an arbitrary continuous function on an interval $[a, b]$ by polynomials.

Theorem 2.17. *Every continuous function on an interval $[a, b]$ can be uniformly approximated by polynomials.*

Proof. Using a linear transformation

$$[a, b] \ni s \mapsto \frac{s-a}{b-a},$$

we see that our task is reduced to the special case when $[a, b] = [0, 1]$. Next, for any continuous function f on $[0, 1]$ let us denote by f_n the n-th Bernstein polynomial defined by

$$f_n(t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) t^k (1-t)^{n-k}, \quad \text{for} \quad t \in [0, 1].$$

We show that f_n converges to f uniformly on $[0,1]$. Let us fix $\varepsilon > 0$. Since the function f is uniformly continuous on $[0,1]$, there exists $\delta > 0$ such that $|f(t) - f(s)| < \varepsilon$ as long as $|t-s| < \delta$. Invoking the identity

$$\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} = 1,$$

we easily get

$$|f(t) - f_n(t)| \le \sum_{k=0}^{n} \binom{n}{k} \left| f(t) - f\left(\frac{k}{n}\right) \right| t^k (1-t)^{n-k}.$$

Then we write

$$\begin{aligned} |f(t) - f_n(t)| &\le \varepsilon \sum_{\substack{k=1 \\ |t-n/k|<\delta}}^{n} \binom{n}{k} t^k (1-t)^{n-k} + 2M \sum_{\substack{k=1 \\ |t-n/k|\ge\delta}}^{n} \binom{n}{k} t^k (1-t)^{n-k} \\ &\le \varepsilon + 2\|f\|_\infty \delta^{-2} \sum_{k=0}^{n} \binom{n}{k} \left(t - \frac{k}{n}\right)^2 t^k (1-t)^{n-k}. \end{aligned}$$

Using the identity

$$\sum_{k=0}^{n} \binom{n}{k} \left(t - \frac{k}{n}\right)^2 t^k (1-t)^{n-k} = \frac{t(1-t)}{n}, \tag{2.18}$$

we get the estimate $\varepsilon + 2\|f\|_\infty \delta^{-2} n^{-1}$. Therefore,

$$\lim_{n\to\infty} \|f - f_n\|_\infty = 0.$$

The proof of Theorem 2.17 is completed. □

Exercise 2.19. Show the following identities

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} t^k s^{n-k} &= (t+s)^n, \\ \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} t^k s^{n-k} &= t(t+s)^{n-1}, \\ \sum_{k=0}^{n} \frac{k^2}{n^2} \binom{n}{k} t^k s^{n-k} &= \frac{t}{n}(t+s)^{n-1} + \frac{n-1}{n} t^2 (t+s)^{n-2}. \end{aligned}$$

Exercise 2.20. Using the previous exercise verify identity (2.18).

Theorem 2.21. *The trigonometric polynomials*

$$a_0 + \sum_{k=1}^{n} \big(a_k \cos kt + b_k \sin kt\big)$$

are dense in the class of all continuous and periodic functions with period 2π.

Proof. Let f be a continuous and periodic function with period 2π. If f is even, then we may treat it as a function on $[0,\pi]$. Further, we may define a continuous function g as follows

$$g(t) = f(\arccos t), \qquad \text{for} \qquad t \in [-1,1].$$

Then $f(t) = g(\cos t)$. Now applying Theorem 2.17 to the function g we see that the conclusion for even functions follows. Assume now that the function f is odd. Therefore the function

$$f_1(t) = f(t)\sin t$$

is even and hence we can approximate it by trigonometric polynomials. This shows that for an arbitrary function f the function $f(t)\sin t$ can be approximated by trigonometric polynomials. Consequently, this means that the same is true for $f(t)\sin^2 t$. Further, we can approximate by trigonometric polynomials the function $f(t)\cos^2 t$. Indeed, we consider $f(\pi/2 - t)$ instead of $f(t)$ and then we change the variable $t \mapsto \pi/2 - t$, which preserves the class of trigonometric polynomials. Since $\sin^2 t + \cos^2 t = 1$, we see that we can approximate f by trigonometric polynomials and the conclusion follows. $\square$

Exercise 2.22. If X is a compact topological space, and $(f_n : n \in \mathbb{N})$ is a monotonically increasing sequence (meaning $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$) of continuous real-valued functions on X which converges pointwise to a continuous function f, then the convergence is uniform. The same conclusion holds if $(f_n : n \in \mathbb{N})$ is monotonically decreasing instead of increasing.

Exercise 2.23. There exists a sequence $(p_n : n \in \mathbb{N})$ of polynomials which is uniformly convergent to the function $\sqrt{x}$ on $[0,1]$. *Hint:* Define $(p_n : n \in \mathbb{N})$ recursively by setting $p_1(x) = 0$ and $p_{n+1}(x) = p_n(x) + \frac{1}{2}(x - p_n(x))^2$ for $n \in \mathbb{N}$ and use the previous problem.

2.3. **Equidistribution.** Here we discuss Weyl's theorem on equidistributed sequences.

Definition 2.24. A sequence $(a_k : k \in \mathbb{N} \cup \{0\}) \subseteq [0,1]$ is called equidistributed if for every function $f \in C([0,1],\mathbb{C})$ we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(a_k) = \int_0^1 f(x)\mathrm{d}x. \tag{2.25}$$

Theorem 2.26. *The following statements are equivalent:*

(a) The sequence $(a_k : k \in \mathbb{N} \cup \{0\}) \subseteq [0,1]$ is equidistributed.

(b) For every $m \in \mathbb{Z} \setminus \{0\}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i m a_k} = 0.$$

(c) For any number a, b with $0 \leq a < b \leq 1$

$$\lim_{N\to\infty} \frac{\#\{k \in [0,N) \cap \mathbb{Z} \colon a_k \in [a,b]\}}{N} = b - a.$$

Proof. We first prove the equivalence of (a) and (c). Assume (a) and fix $0 \leq a < b \leq 1$. Given a sufficiently small $\varepsilon > 0$, we define continuous functions $f^-, f^+ : [0,1] \to [0,1]$ that approximate the indicator function $\mathbb{1}_{[a,b]}$ by

$$f^+(x) = \begin{cases} 1 & \text{if } a \leq x \leq b; \\ \varepsilon^{-1}(x - (a - \varepsilon)) & \text{if } \max\{0, a-\varepsilon\} \leq x < a; \\ \varepsilon^{-1}((b+\varepsilon) - x) & \text{if } b < x \leq \max\{b+\varepsilon, 1\}; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f^-(x) = \begin{cases} 1 & \text{if } a+\varepsilon \le x \le b-\varepsilon; \\ \varepsilon^{-1}(x-a) & \text{if } a \le x < a+\varepsilon; \\ \varepsilon^{-1}(b-x) & \text{if } b-\varepsilon < x \le b; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $f^-(x) \le \mathbb{1}_{[a,b]}(x) \le f^+(x)$ for all $x \in [0,1]$, and

$$\int_0^1 (f^+(x) - f^-(x))\mathrm{d}x \le 2\varepsilon.$$

It follows that

$$\frac{1}{N}\sum_{k=0}^{N-1} f^-(a_k) \le \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) \le \frac{1}{N}\sum_{k=0}^{N-1} f^+(a_k).$$

By equidistribution this implies that

$$\begin{aligned} b-a-2\varepsilon \le \int_0^1 f^-(x)\mathrm{d}x \le \liminf_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) \\ \le \limsup_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) \le \int_0^1 f^+(x)\mathrm{d}x \le b-a+2\varepsilon. \end{aligned}$$

Thus (c) is proved

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) = \limsup_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) = b-a.$$

Assume that (c) holds. Given a continuous function f on $[0,1]$ and given $\varepsilon > 0$, we find a step function $g = \sum_{j=1}^m c_j \mathbb{1}_{I_j}$ ($c_j \in \mathbb{C}$ and intervals $I_j \subseteq [0,1]$) such that $\|f-g\|_\infty < \varepsilon/3$. Since g is a finite linear combination of step functions, there is an N_0 such that for $N \ge N_0$ we have

$$\Big|\frac{1}{N}\sum_{k=0}^{N-1} g(a_k) - \int_0^1 g(x)\mathrm{d}x\Big| < \varepsilon/3.$$

Since

$$\Big|\int_0^1 f(x)\mathrm{d}x - \int_0^1 g(x)\mathrm{d}x\Big| \le \|f-g\|_\infty < \varepsilon/3$$

and

$$\Big|\frac{1}{N}\sum_{k=0}^{N-1} g(a_k) - \frac{1}{N}\sum_{k=0}^{N-1} f(a_k)\Big| \le \|f-g\|_\infty < \varepsilon/3,$$

it follows that for $N \ge N_0$ we have

$$\Big|\frac{1}{N}\sum_{k=0}^{N-1} f(a_k) - \int_0^1 f(x)\mathrm{d}x\Big| < \varepsilon,$$

thus (a) holds.

We now prove the equivalence of (a) and (b). In one direction this is clear. To see that (b) implies (a) we fix a continuous function f on $[0,1]$. Then for a given $\varepsilon > 0$ by Theorem 2.21 we pick a trigonometric polynomial p such that $\|f-p\|_\infty < \varepsilon/3$. Since

$$p(x) = \sum_{m=-M}^{M} c_m e^{2\pi i m x}$$

for some $M \in \mathbb{N}$ and $c_m \in \mathbb{C}$, then by (b) we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} p(a_k) = c_0 = \int_0^1 p(x)\mathrm{d}x.$$

Hence a 3-epsilon argument as above yields

$$\Big|\frac{1}{N}\sum_{k=0}^{N-1} f(a_k) - \int_0^1 f(x)\mathrm{d}x\Big| < \varepsilon,$$

and the proof is completed. □

Example 2.27. The sequence $(\{k\sqrt{2}\} : k \in \mathbb{N}\cup\{0\})$ is equidistributed on $[0,1]$. We check this by verifying condition (b) of Theorem 2.26. Indeed if $m \in \mathbb{Z}\setminus\{0\}$ then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i m(k\sqrt{2}-\lfloor k\sqrt{2}\rfloor)} = \lim_{N\to\infty}\frac{1}{N}\frac{e^{2\pi i N m\sqrt{2}}-1}{e^{2\pi i m\sqrt{2}}-1} = 0,$$

since $m\sqrt{2}$ is never a rational and thus the denominator never vanishes.

Naturally, the same conclusion is valid for any other irrational number $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ in place of $\sqrt{2}$.

Example 2.28. We now return to the second part of Gelfand's question. We recall that for $m \in \mathbb{N}$ we consider

$$d_m = \text{first digit of } 2^m.$$

Fix an integer $1 \le k \le 9$. We will find the frequency in which k appears as a first digit of 2^m, precisely, we would like to compute

$$\lim_{N\to\infty} \frac{\#\{m \in \{1,\ldots,N\} : d_m = k\}}{N}.$$

As we mentioned above it is essential that the first digit of 2^m is equal to k if and only if there is a nonnegative integer s such that

$$k10^s \le 2^m < (k+1)10^s.$$

Taking logarithms with base 10 we obtain

$$s + \log_{10} k \le m\log_{10} 2 < s + \log_{10}(k+1),$$

but since $0 \le \log_{10} k$ and $\log_{10}(k+1) \le 1$, taking fractional parts we obtain that

$$s = \lfloor m\log_{10} 2\rfloor$$

and that

$$\log_{10} k \le m\log_{10} 2 - \lfloor m\log_{10} 2\rfloor < \log_{10}(k+1).$$

Since the number $\log_{10} 2$ is irrational, it follows from Example 2.27 that the sequence

$$\big(\{m\log_{10} 2\} : m \in \mathbb{N}\big)$$

is equidistributed in $[0,1]$. Using (c) from Theorem 2.26 with $[a,b] = [\log_{10} k, \log_{10}(k+1)]$ we obtain that

$$\lim_{N\to\infty} \frac{\#\{m \in \{1,\dots,N\} : d_m = k\}}{N} = \log_{10}(k+1) - \log_{10} k = \log_{10}(1+1/k).$$

This gives the frequency in which k appears as first digit of 2^m. Notice that

$$\sum_{k=1}^{9} \log_{10}(1+1/k) = 1,$$

as expected. Moreover, the digit with the highest frequency that appears as the first digit in the decimal expansion of the sequence $(2^m : m \in \mathbb{N})$ is 1, while the one with the lowest frequency is 9.

Exercise 2.29 (Kronecker)**.** Suppose that $N \in \mathbb{N}$ and

$$\{x_1, x_2, \dots, x_N, 1\}$$

is a linearly independent set over the rationals. Prove that for any $\varepsilon > 0$ and any complex numbers $z_1, z_2, \dots, z_N$ with $|z_j| = 1$, there exists an integer $L \in \mathbb{Z}$ such that

$$|e^{2\pi i L x_j} - z_j| < \varepsilon, \qquad \text{for all} \quad 1 \le j \le N.$$

Exercise 2.30. Suppose that $(X, \mathcal{M}, \mu)$ be a finite measure space, $f \in L^1(X,\mu)$, $S \subseteq \mathbb{C}$ is a closed set, and the averages

$$A_E(f) = \frac{1}{\mu(E)} \int_E f(x) \mathrm{d}\mu(x) \in S$$

for every $E \in \mathcal{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost every $x \in X$.

Exercise 2.31. If $z_1, \dots, z_N$ are complex numbers then there is a subset S of $\{1,\dots,N\}$ for which

$$\Big|\sum_{k\in S} z_k\Big| \ge \frac{1}{\pi} \sum_{k=1}^{N} |z_k|.$$

3. Algebraic and transcendental numbers

This section is based on [6] and [1].

3.1. Liouville numbers.

If one can show that the set of numbers in an interval that lack a certain property is either countable, or a nullset, or a set of the first category, then it follows that there exist points of the interval that have the property in question, in fact, most points of the interval (in the sense of cardinal number, or measure, or category, respectively) have the property. This method in the context of category has been presented in the first section. Using the same kind of ideas one can prove the following:

Exercise 3.1. Show that not every Lebesgue measurable subset of $\mathbb{R}$ is a Borel set. *Hint:* Use the ternary Cantor set and the fact that the Lebesgue measure is complete.

Another illustration of this method will be the existence of transcendental numbers. A complex number z is called algebraic if it satisfies some equation of the form

$$a_n z^n + \ldots + a_1 z + a_0 = 0$$

with integer coefficients, not all zero. The degree of an algebraic number z is the smallest positive integer n such that z satisfies an equation of degree n. For instance, any rational number is algebraic of degree 1, $\sqrt{2}$ is algebraic of degree 2, and $\sqrt{2}+\sqrt{3}$ is algebraic of degree 4. Any real number that is not algebraic is called transcendental. Do there exist transcendental numbers?

Exercise 3.2. Show that the set of real algebraic numbers is countable.

Exercise 3.2 gives perhaps the simplest proof of the existence of transcendental numbers. It should be noted that it is not an indirect proof. But due to its existential nature we know nothing concrete about transcendental numbers. An older and more informative proof of the existence of transcendental numbers is due to Liouville. His proof is based on the following:

Proposition 3.3. *For any real algebraic number z of degree $n > 1$ there exists a positive integer M such that*

$$\left| z - \frac{p}{q} \right| > \frac{1}{Mq^n} \tag{3.4}$$

for all $p, q \in \mathbb{Z}$ with $q > 0$.

Proof. Let $f(x)$ be a polynomial of degree n with integer coefficients for which $f(z) = 0$. Let M be a positive integer such that $|f'(x)| \leq M$ whenever $|z - x| \leq 1$. Then, by the mean value theorem,

$$|f(x)| = |f(x) - f(z)| \leq M|x - z| \qquad \text{whenever} \qquad |z - x| \leq 1. \tag{3.5}$$

Now consider any two integers p and q, with $q > 0$. We wish to show (3.4). This is evidently true in case $|z - p/q| > 1$, so we may assume that $|z - p/q| \leq 1$. Then, by (3.5), we have $|f(p/q)| \leq M|z - p/q|$ and therefore

$$\left| q^n f\left(\frac{p}{q}\right) \right| \leq Mq^n \left| z - \frac{p}{q} \right|. \tag{3.6}$$

The equation $f(x) = 0$ has no rational root (otherwise z would satisfy an equation of degree less than n). Moreover, $q^n f(p/q)$ is an integer. Hence the left hand-side of (3.6) is at least 1 and we conclude that (3.4) must hold. Equality cannot hold, because z is irrational. □

Proposition 3.7. *The number*

$$a = \sum_{n=1}^{\infty} \frac{1}{2^{n!}}$$

is transcendental.

Proof. It is not difficult to show that $a \notin \mathbb{Q}$. Suppose that a is an algebraic number of degree $N > 1$. Then by Proposition 3.3 we can find an integer $M \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ we have

$$\sum_{n=k+1}^{\infty} \frac{1}{2^{n!}} = \left| a - \sum_{n=1}^{k} \frac{1}{2^{n!}} \right| \geq \frac{1}{M 2^{k!N}},$$

since $\sum_{n=1}^{k} \frac{1}{2^{n!}}$ is a number with denominator $2^{k!}$. On the other hand, we have

$$\sum_{n=k+1}^{\infty} \frac{1}{2^{n!}} \leq \frac{2}{2^{(k+1)!}},$$

hence

$$\frac{1}{M 2^{k!N}} \leq \frac{2}{2^{(k+1)!}} \qquad \Longleftrightarrow \qquad 2^{k!(1+k-N)} \leq 2M,$$

which is impossible for sufficiently large $k \in \mathbb{N}$. □

A real number z is called a Liouville number if z is irrational and has the property that for each positive integer n there exist integers p and q such that

$$\left|z - \frac{p}{q}\right| < \frac{1}{q^n} \quad \text{and} \quad q > 1.$$

Theorem 3.8 (Liouville)**.** *Every Liouville number is transcendental.*

Proof. Suppose some Liouville number z is algebraic, of degree n. Then $n > 1$, since z is irrational. By Proposition 3.3 there exists a positive integer M such that

$$\left|z - \frac{p}{q}\right| > \frac{1}{Mq^n} \tag{3.9}$$

for all $p, q \in \mathbb{N}$ with $q > 0$. Choose a positive integer k such that $2^k \geq 2^n M$. Because z is a Liouville number there exist integers p and q, with $q > 1$, such that

$$\left|z - \frac{p}{q}\right| < \frac{1}{q^k}. \tag{3.10}$$

Combining (3.9) and (3.10) it follows that $1/q^k > 1/(Mq^n)$. Hence $M > q^{k-n} \geq 2^{k-n} \geq M$ a contradiction. □

Let us examine the set E of Liouville numbers. From the definition it follows at once that

$$E = \mathbb{R} \setminus \mathbb{Q} \cap \bigcap_{n \in \mathbb{N}} G_n, \tag{3.11}$$

where

$$G_n = \bigcup_{q \geq 2} \bigcup_{p \in \mathbb{Z}} (p/q - 1/q^n, p/q + 1/q^n)$$

is a union of open intervals. Moreover, G_n includes every number of the form p/q, with $q \geq 2$, hence $\mathbb{Q} \subseteq G_n$. Therefore G_n is a dense open set, and so its complement is nowhere dense. Since, by (3.11), we have

$$E^c = \mathbb{Q} \cup \bigcup_{n \in \mathbb{N}} G_n^c.$$

It follows that E^c is of the first category. Thus Baire's theorem implies that Liouville transcendental numbers exist in every interval, they are "generic" in the sense of category.

What about the measure of E? From (3.11) it follows that $E \subseteq G_n$ for every $n \in \mathbb{N}$. For any integer $q \geq 2$ let

$$G_{n,q} = \bigcup_{p \in \mathbb{Z}} (p/q - 1/q^n, p/q + 1/q^n).$$

For any two positive integers m and n we have

$$\begin{aligned} E \cap (-m, m) &\subseteq G_n \cap (-m, m) \\ = \bigcup_{q \geq 2} G_{n,q} \cap (-m, m) &\subseteq \bigcup_{q \geq 2} \bigcup_{p=-mq}^{mq} (p/q - 1/q^n, p/q + 1/q^n). \end{aligned}$$

Therefore $E \cap (-m, m)$ can be covered by a sequence of intervals such that for any $n > 2$, we have

$$\begin{aligned}\sum_{q\geq 2}\sum_{p=-mq}^{mq} 2/q^n &= \sum_{q\geq 2}(2mq+1)(2/q^n)\\ &\leq \sum_{q\geq 2}(4mq+q)(1/q^n)\\ &= (4m+1)\sum_{q\geq 2}(1/q^{n-1})\\ &\leq \frac{4m+1}{n-2}.\end{aligned}$$

It follows that $E \cap (-m, m)$ is a nullset for every m, and therefore E is a nullset. Thus E is small in the sense of measure, but large in the sense of category. The sets E and E^c provide a decomposition of the line into a set of measure zero and a set of the first category.

Exercise 3.12. The construction of the Cantor set by starting with $[0, 1]$ and successively removing open middle thirds of intervals has an obvious generalization. If I is a bounded interval and $\alpha \in (0, 1)$, let us call the open interval with the same midpoint as I and length equal to α times the length of I the „open middle αth" of I. If $(\alpha_j)_{j\in\mathbb{N}}$ is any sequence of numbers in $(0, 1)$, then, we can define a decreasing sequence $(K_j)_{j\geq 0}$ of closed sets as follows: $K_0 = [0, 1]$, and K_j is obtained by removing the open middle α_jth from each of the intervals that make up K_{j-1}. The resulting limiting set $K = \bigcap_{j\in\mathbb{N}} K_j$ is called a generalized Cantor set. Show that

(a) K is compact, nowhere dense and has no isolated points.
(b) Cardinality of K is $\mathfrak{c}$.
(c) If $\alpha_j = \frac{1}{3}$ for all $j \in \mathbb{N}$ then we obtain the ordinary ternary Cantor set, and it has the Lebesgue measure zero.
(d) If $\alpha_j \xrightarrow[j\to\infty]{} 0$ then the Lebesgue measure of K will be positive and in fact for any $\beta \in (0, 1)$ one can choose α_j so that the Lebesgue measure will be β.

Exercise 3.13. Consider $\mathbb{R}$ with the Lebesgue measure λ. Construct a set $X \subseteq \mathbb{R}$ such that for every open non-empsty set $V \subseteq \mathbb{R}$ we have

$$\lambda(V \cap X) > 0 \qquad \text{and} \qquad \lambda(V \cap X^c) > 0.$$

Exercise 3.14. Let λ be the usual measure on $X = \{0, 1\}^{\mathbb{N}}$, and let μ be Lebesgue measure on $[0, 1]$; write Λ for the domain of λ and Σ for the domain of μ.

(a) For $x \in X$ set $\phi(x) = \sum_{i=0}^{\infty} 2^{-i-1}x(i)$. Then
  (i) $\phi^{-1}[E] \in \Lambda$ and $\lambda(\phi^{-1}[E]) = \mu(E)$ for every $E \in \Sigma$;
  (ii) $\phi[F] \in \Sigma$ and $\mu(\phi[F]) = \lambda(F)$ for every $F \in \Lambda$.
(b) There is a bijection $\tilde{\phi} : X \to [0, 1]$ which is equal to ϕ at all but countably many points, and any such bijection is an isomorphism between (X, Λ, λ) and $([0, 1], \Sigma, \mu)$.

3.2. **Hermite's theorem.** Squaring the circle is a problem proposed by ancient geometers. It was the challenge of constructing a square with the same area as a given circle by using only a finite number of steps with compass and straightedge. In 1882, the task was proven to be impossible, as a consequence of the Lindemann–Weierstrass theorem which proves that π is a transcendental, rather than an algebraic irrational number. It had been known for some decades before then that the construction would be impossible if π were transcendental, but π was not proven transcendental until 1882. A bit simpler is to show that e is transcendental. Before we do this we show that e is irrational.

Proposition 3.15. *The number*

$$e = \sum_{k \geq 0} \frac{1}{k!}$$

is irrational.

Proof. To start with, it is rather easy to see (as did Fourier in 1815) that $e = \sum_{k \geq 0} 1/k!$ is irrational. Indeed, if we had $e = a/b$ for integers a and $b > 0$, then we would get

$$n!be = n!a$$

for every integer $n \geq 0$. But this cannot be true, because on the right-hand side we have an integer, while the left-hand side with

$$e = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) + \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots\right)$$

decomposes into an integral part

$$n!b\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right)$$

and a second part

$$b\left(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \ldots\right)$$

which is approximately $\frac{b}{n}$, so that for large n it certainly cannot be integral. It is larger than $\frac{b}{n+1}$ and smaller than $\frac{b}{n}$, as one can see from a comparison with a geometric series

$$\begin{aligned}\frac{1}{n+1} &< \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \ldots \\ &< \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots < \frac{1}{n}.\end{aligned}$$

We get a contradiction and the proof is completed. □

Theorem 3.16 (Hermite's theorem)**.** *The number*

$$e = \sum_{k \geq 0} \frac{1}{k!}$$

is transcendental.

Proof. If e were an algebraic number, then we could find a polynomial P with rational coefficients such that

$$P(x) = a_n x^n + \ldots + a_1 x + a_0$$

satisfying $P(e) = 0$. For every prime number $p \in \mathbb{P}$ satisfying $p > n$ and $p > |a_0|$ we define an auxiliary polynomial by setting

$$f_p(x) = \frac{x^{p-1}}{(p-1)!} \prod_{k=1}^{n} (k - x)^p$$

and we also set

$$F_p(x) = f_p(x) + \sum_{j=1}^{M} f_p^{(j)}(x),$$

where $M = (n+1)p - 1$ is the degree of the polynomial f_p. Since $f_p^{(M+1)}(x) = 0$ we obtain

$$F_p(x) - F_p'(x) = f_p(x),$$

and consequently

$$\big(e^{-x}F_p(x)\big)' = -e^{-x}F_p(x) + e^{-x}F_p'(x) = -e^{-x}f_p(x).$$

By the mean-value theorem we get

$$e^{-x}F_p(x) - F_p(0) = -xe^{-\theta_x x}f_p(\theta_x x)$$

for some $\theta_x \in [0,1]$. Thus

$$F_p(x) - e^x F_p(0) = -xe^{(1-\theta_x)x}f_p(\theta_x x).$$

If x is fixed and $p \to \infty$, then

$$\lim_{p\to\infty}\big(F_p(x) - e^x F_p(0)\big) = 0,$$

since for every $y \in \mathbb{R}$ we have $\lim_{n\to\infty}\frac{y^n}{n!} = 0$. We also obtain

$$\lim_{p\to\infty}\sum_{k=0}^{n} a_k F_p(k) = \lim_{p\to\infty}\left(\sum_{k=0}^{n} a_k F_p(k) - F_p(0)\sum_{k=0}^{n} a_k e^k\right) = 0. \tag{3.17}$$

Since $j!$ divides all coefficients of j-th derivative of an arbitrary polynomial we get for a suitable polynomials P_j with integer coefficients that

$$f_p^{(j)}(x) = \frac{j!}{(p-1)!}P_j(x).$$

Hence,

$$F_p(0) = \sum_{j=p-1} f_p^{(j)}(0) = \frac{1}{(p-1)!}\sum_{j=p-1} j!P_j(0) \equiv P_{p-1}(0)(\mathrm{mod}\, p),$$

since

$$f_p(0) = f_p'(0) = \ldots = f_p^{(p-2)}(0) = 0.$$

Moreover all numbers $\frac{1}{(p-1)!}\sum_{j=p-1} j!P_j(0) \in \mathbb{Z}$ and are divisible by p. Similarly, for $f_p^{(i)}(k) = 0$ for $i \in \{1,\ldots,p-1\}$ and $k \in \{1,\ldots,n\}$, thus

$$F_p(k) = \sum_{j=p} f_p^{(j)}(k) = \frac{1}{(p-1)!}\sum_{j=p} j!P_j(k) \equiv 0(\mathrm{mod}\, p).$$

Finally,

$$\sum_{k=0}^{n} a_k F_p(k) \equiv a_0F_p(0) \equiv a_0P_{p-1}(0) \equiv a_0(n!)^p \not\equiv 0(\mathrm{mod}\, p).$$

This contradicts with (3.17), since a sequence of integer that converges to 0 must be constant for all but finitely many terms. This completes the proof of theorem. □

Remark 3.18. We now know that e is transcendental. In 1882 Lindemann, following the ideas from Hermite's proof, showed that π is transcendental as well. For example, it is unknown whether $\pi+e$ is transcendental, though at least one of $\pi+e$ and πe must be transcendental. From Theorem 3.8 we know that all Liouville numbers are transcendental, but not vice versa. Any Liouville number must have unbounded partial quotients in its continued fraction expansion. Using a counting argument one can show that there exist transcendental numbers which have bounded partial quotients and hence are not Liouville numbers.

Using the explicit continued fraction expansion of e, one can show that e is not a Liouville number (although the partial quotients in its continued fraction expansion are unbounded). Kurt Mahler showed in 1953 that π is also not a Liouville number. It is conjectured that all infinite continued fractions with bounded terms that are not eventually periodic are transcendental.

Our aim will be to show that for every $m \in \mathbb{N}$ the number

$$\sum_{n\in\mathbb{N}} \frac{1}{n^{2m}}$$

is transcendental. In other words, the Riemann zeta function $\zeta(s)$ evaluated at even integers is a transcendental number. We will obtain this result by explicit computations.

3.3. **Euler's series.** We shall show the following:

Proposition 3.19. *One has*

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{3.20}$$

This is a classical, famous and important result by Leonhard Euler from 1734. One of its key interpretations is that it yields the first non-trivial value $\zeta(2)$ of Riemann's zeta function.

Proof. The proof consists in two different evaluations of the double integral

$$I = \int_0^1 \int_0^1 \frac{1}{1-xy} \mathrm{d}x\mathrm{d}y.$$

For the first one, we expand $\frac{1}{1-xy}$ as a geometric series, decompose the summands as products, and integrate effortlessly:

$$\begin{aligned} I &= \int_0^1 \int_0^1 \sum_{n\geq 0} (xy)^n \mathrm{d}x\mathrm{d}y \\ &= \sum_{n\geq 0} \int_0^1 \int_0^1 (xy)^n \mathrm{d}x\mathrm{d}y \\ &= \sum_{n\geq 0} \frac{1}{n+1}\frac{1}{n+1} \\ &= \sum_{n\in\mathbb{N}} \frac{1}{n^2} = \zeta(2). \end{aligned}$$

This evaluation also shows that the double integral (over a positive function with a pole at $x = y = 1$) is finite. Note that the computation is also easy and straightforward if we read it backwards – thus the evaluation of $\zeta(2)$ leads one to the double integral I.

The second way to evaluate I comes from a change of coordinates: in the new coordinates given by $u := \frac{y+x}{2}$ and $v := \frac{y-x}{2}$ the domain of integration is a square of side length $\frac{1}{2}\sqrt{2}$, which we get from the old domain by first rotating it by 45 degree and then shrinking it by a factor of $\sqrt{2}$. Substitution of $x = u - v$ and $y = u + v$ yields

$$\frac{1}{1-xy} = \frac{1}{1-u^2+v^2}.$$

To transform the integral, we have to replace $\mathrm{d}x\mathrm{d}y$ by $2\mathrm{d}u\mathrm{d}v$, to compensate for the fact that our coordinate transformation reduces areas by a constant factor of 2 (which is the Jacobi determinant of the transformation). The new domain of integration, and the function to be integrated, are symmetric with respect to the u-axis, so we just need to compute two times (another factor of 2 arises here!) the integral over the upper half domain, which we split into two parts in the most natural way:

$$I = 4\int_0^{1/2}\left(\int_0^u \frac{\mathrm{d}v}{1-u^2+v^2}\right)\mathrm{d}u + 4\int_{1/2}^{1}\left(\int_0^{1-u} \frac{\mathrm{d}v}{1-u^2+v^2}\right)\mathrm{d}u.$$

Using

$$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C,$$

this becomes

$$\begin{aligned} I =& 4\int_0^{1/2} \frac{1}{\sqrt{1-u^2}} \arctan\left(\frac{u}{\sqrt{1-u^2}}\right)\mathrm{d}u \\ &+ 4\int_{1/2}^{1} \frac{1}{\sqrt{1-u^2}} \arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right)\mathrm{d}u. \end{aligned}$$

Observe that taking

$$g(u) = \arctan\left(\frac{u}{\sqrt{1-u^2}}\right), \qquad \text{we have} \qquad g'(u) = \frac{1}{\sqrt{1-u^2}},$$

while for

$$h(u) = \arctan\left(\frac{1-u}{\sqrt{1-u^2}}\right), \qquad \text{we have} \qquad h'(u) = -\frac{1}{2}\frac{1}{\sqrt{1-u^2}}.$$

Using

$$\int_a^b f'(x)f(x)\mathrm{d}x = \left[\frac{1}{2}f(x)^2\right]\Bigg|_a^b = \frac{1}{2}\big(f(b)^2 - f(a)^2\big),$$

we obtain that

$$\begin{aligned} I &= 2\int_0^{1/2} 2g'(u)g(u)\mathrm{d}u - 4\int_{1/2}^1 2h'(u)h(u)\mathrm{d}u \\ &= 2\big[g(u)^2\big]\Big|_0^{1/2} - 4\big[h(u)^2\big]\Big|_{1/2}^1 \\ &= 2g(1/2)^2 - 2g(0)^2 - 4h(1)^2 + 4h(1/2)^2 \\ &= 2\left(\frac{\pi}{6}\right)^2 - 0 - 0 + 4\left(\frac{\pi}{6}\right)^2 \\ &= \frac{\pi^2}{6}, \end{aligned}$$

and we are done. □

3.4. **The Bernoulli numbers.** The Bernoulli numbers arise naturally in the context of computing the power sums

$$\begin{aligned} 1^0 + 2^0 + \ldots + n^0 &= n, \\ 1^1 + 2^1 + \ldots + n^1 &= \frac{1}{2}(n^2 + n), \\ 1^2 + 2^2 + \ldots + n^2 &= \frac{1}{6}(2n^3 + 3n^2 + n), \\ 1^3 + 2^3 + \ldots + n^3 &= \frac{1}{4}(n^4 + 2n^3 + n^2), \\ &\text{etc.} \end{aligned}$$

To study these, let n be a positive integer and let the k-th power sum up to $n-1$ be

$$S_k(n) = \sum_{m=0}^{n-1} m^k \quad \text{for} \quad k \in \mathbb{N}.$$

Thus $S_0(n) = n$ while for $k > 0$ the term 0^k is 0. (Having the sum start at 0 and stop at $n-1$ neatens the ensuing calculation.) The power series having these sums as its coefficients is their generating function,

$$\mathbf{S}(n,t) = \sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!}.$$

Exercise 3.21. Show that

$$\mathbf{S}(n,t) = \frac{e^{nt}-1}{t}\,\frac{t}{e^t-1}.$$

The second term is independent of n. Its coefficients are by definition the Bernoulli numbers, constants B_k that can be computed once and for all,

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k\frac{t^k}{k!}.$$

Exercise 3.22. Show that the generating function rearranges to

$$\begin{aligned}\mathbf{S}(n,t) &= \sum_{m=1}^{\infty} n^m \frac{t^{m-1}}{m!} \sum_{j=0}^{\infty} B_j \frac{t^j}{j!} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} \right) \frac{t^k}{k!}.\end{aligned}$$

Thus, if we define the kth Bernoulli polynomial as

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_j x^{k-j},$$

which again can be computed once and for all.

Exercise 3.23. Show that

$$S_k(n) = \frac{1}{k+1}\big(B_{k+1}(n) - B_{k+1}\big).$$

The first few Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$ and so the first few Bernoulli polynomials are

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x + \frac{1}{6}, \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x. \end{aligned}$$

Exercise 3.24. Show for every $k \in \mathbb{N}$ that

$$B_k'(x) = kB_{k-1}(x),$$

assuming that $B_0(x) = 1$, and

$$\int_0^1 B_k(x)\mathrm{d}x = 0 \quad \text{for any} \quad k \in \mathbb{N}.$$

3.5. **Weierstrass theorem for complex functions.** We shall prove a result which guarantees that any complex power series can be differentiated term-by-term within its disk of convergence. The result will have other consequences as well.

Exercise 3.25. Let S be a subset of $\mathbb{C}$. Consider a sequence of continuous functions on S,

$$\varphi_1, \varphi_2, \ldots : S \to \mathbb{C}.$$

Suppose that the sequence converges uniformly on S to a limit function

$$\varphi \colon S \to \mathbb{C}.$$

Then φ is also continuous.

The other result to recall is that we can pass uniform limits through integrals.

Exercise 3.26. Let Ω be a region in $\mathbb{C}$ and let a be any point of Ω. Suppose that a closed ball B centered at a lies in Ω. Let $\gamma = \partial B$ be the boundary circle of B, traversed once counterclockwise. Suppose that a sequence of continuous functions

$$\phi_1, \phi_2, \ldots : \gamma \to \mathbb{C}$$

converges uniformly on γ to a limit $\phi \colon \gamma \to \mathbb{C}$. Then

$$\lim_{n\to\infty} \int_\gamma \phi_n(z)\mathrm{d}z = \int_\gamma \phi(z)\mathrm{d}z.$$

Now we can state and prove our main result.

Theorem 3.27 (Weierstrass)**.** *Let Ω be a region in $\mathbb{C}$. Consider a sequence of differentiable functions on Ω,*

$$\varphi_1, \varphi_2, \ldots : \Omega \to \mathbb{C}.$$

Suppose that the sequence converges on Ω to a limit function

$$\varphi \colon \Omega \to \mathbb{C}.$$

and that the convergence is uniform on compact subsets of Ω. Then

(1) The limit function φ is differentiable.
(2) One has $\lim_{n\to\infty} \varphi_n'(z) = \varphi'(z)$ for every $z \in \Omega$.
(3) This convergence is also uniform on compact subsets of Ω.

Proof. First, to show that the limit function φ is continuous, let z be any point of Ω. Some closed ball B centered at z lies in Ω, and the convergence of $(\varphi_n : n \in \mathbb{N})$ to φ is uniform on the compact set B. The restriction of the limit function φ to B is therefore continuous, and so φ itself is continuous at the interior point z. Since z is arbitrary, φ is continuous on Ω.

Next, to show that φ is differentiable, let $\gamma = \partial B$ be the boundary circle of the closed ball B from the previous paragraph, traversed once counterclockwise, and let z be any point inside γ. Consider an auxiliary sequence of functions on γ, defined by

$$\phi_n(\zeta) = \frac{\varphi_n(\zeta)}{\zeta - z}, \qquad n \in \mathbb{N}$$

with limit

$$\phi(\zeta) = \frac{\varphi(\zeta)}{\zeta - z}.$$

Since ϕ_n converges uniformly on γ to ϕ, we may exchange an integral and a limit,

$$\begin{aligned}
\varphi(z) &= \lim_{n\to\infty} \varphi_n(z) \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{\varphi_n(\zeta)}{\zeta - z}\mathrm{d}\zeta \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \phi_n(\zeta)\mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \phi(\zeta)\mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\zeta)}{\zeta - z}\mathrm{d}\zeta.
\end{aligned}$$

This shows that the continuous function φ has a Cauchy integral representation, making it differentiable.

Third, use the Cauchy integral representation of derivatives to argue similarly (with a modified auxiliary sequence $(\phi_n : n \in \mathbb{N})$) that the sequence $(\varphi_n' : n \in \mathbb{N})$ of derivatives converges to the derivative φ' of the limit function,

$$\begin{aligned}\lim_{n\to\infty} \varphi_n'(z) &= \lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{\varphi_n(\zeta)}{(\zeta-z)^2}\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\zeta)}{(\zeta-z)^2}\mathrm{d}\zeta \\ &= \varphi'(z).\end{aligned}$$

Finally, we need to argue that this convergence is uniform on compact subsets of Ω. In the special case that the compact set is the closed ball B' having half the radius of the open ball B, let $c > 0$ denote the half-radius, so that

$$|\zeta - z| \geq c, \qquad \zeta \in \gamma, \quad z \in B'.$$

It follows by Cauchy's formula for the derivative and the usual estimation techniques that for all $z \in B'$,

$$\begin{aligned}|\varphi'(z) - \varphi_n'(z)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{\varphi(\zeta)-\varphi_n(\zeta)}{(\zeta-z)^2}\mathrm{d}\zeta\right| \\ &\leq \frac{1}{2\pi}\int_\gamma \frac{|\varphi(\zeta)-\varphi_n(\zeta)|}{c^2}|\mathrm{d}\zeta| \\ &= C\sup_{\zeta\in\gamma}|\varphi(\zeta)-\varphi_n(\zeta)|.\end{aligned}$$

But φ_n converges to φ uniformly on the compact subset γ of Ω. So, given $\varepsilon > 0$, there exists a starting index $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow |\varphi'(z) - \varphi_n'(z)| < \varepsilon \quad \text{ for any } \quad z \in B'.$$

To complete the argument, let K be any compact set of the whole region Ω. About each point a of K there is an open ball $B = B_a$. Let B_a' be the ball about a of half the radius of B_a. These balls give an open cover of K,

$$K = \bigcup_{a\in K}\{a\} \subseteq \bigcup_{a\in K} B_a'.$$

By compactness, K has a finite subcover,

$$K \subseteq B_{a_1}' \cup \ldots \cup B_{a_k}'.$$

Let $K_j = K \cap \mathrm{cl}(B_{a_j}')$ for $j \in \{1, \ldots, k\}$. Then

$$K = K_1 \cup \ldots \cup K_n,$$

and by the previous paragraph, the convergence of $(\varphi_n' : n \in \mathbb{N})$ to φ' is uniform on each of the finitely-many sets K_j. Consequently the convergence is uniform on K: Given $\varepsilon > 0$, the corresponding global starting index $n_0 \in \mathbb{N}$ is the maximum of finitely many local ones. This completes the proof. $\square$

Example 3.28. As mentioned, the application of the Weierstrass theorem that we have in mind here is that the functions φ_n are the partial sums of a power series,

$$\varphi_n(z) = \sum_{j=0}^{n} a_j(z-c)^j, \qquad n \in \mathbb{N} \cup \{0\}$$

while φ is the full power series,

$$\varphi(z) = \sum_{j=0}^{\infty} a_j(z-c)^j.$$

In this case, the result is that any power series can be differentiated term by term within its disk of convergence, and the resulting power series has the same disk of convergence.

Example 3.29. For another application of the Weierstrass theorem, consider the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Since $|1/n^s| = 1/n^{\Re s}$, the sum converges absolutely on the right half plane $\Omega = \{s \in \mathbb{C} : \Re s > 1\}$, and the convergence is uniform on compacta. Thus $\zeta(s)$ is analytic on Ω.

3.6. **Euler's cotangent representation.** We shall establish the following formula

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}.$$

The function $\pi \cot \pi z$ (for nonintegers $z \in \mathbb{C}$) is analytic and $\mathbb{Z}$-periodic. Near $z = 0$ we have

$$\pi \cot \pi z \sim \pi \frac{1}{\pi z} = \frac{1}{z},$$

so that $\pi \cot \pi z$ is also meromorphic at 0, having a simple pole there with residue 1. By $\mathbb{Z}$-perodicity, the same holds at each integer n. Thus, a naive first attempt to imitate $\pi \cot \pi z$ by a series is

$$\sum_{n \in \mathbb{Z}} \frac{1}{z-n}.$$

However, the nth term of this series is $O(1/n)$, so that the series is not even summable. One can fix this problem by modifying the terms to obtain the series

$$\frac{1}{z} + \sum_{n \neq 0} \Big(\frac{1}{z-n} + \frac{1}{n} \Big).$$

Now the nth term is

$$\frac{1}{z-n} + \frac{1}{n} = \frac{z}{(z-n)n} = O(n^{-2}),$$

and so the new series is summable. In fact, this calculation shows that the new series is absolutely summable, so that its terms can be rearranaged. In particular, pairing the terms for n and $-n$ gives

$$\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z+n} - \frac{1}{n} = \frac{1}{z-n} + \frac{1}{z+n} = \frac{2z}{z^2 - n^2},$$

and these are the terms of the series that we began with, in both of its forms. So at least that series converges absolutely for any noninteger $z \in \mathbb{C}$.

All of this said, the series that we began with (in either of its forms) is not a Laurent series, and so part of the task here is to show that it defines a meromorphic function at all. And even if it does, the preceding calculation has exposed a problem. The nth term-with-correction of the series, evaluated at $z+m$ (where m is an integer) rather than at z, is

$$\frac{1}{z+m-n}-\frac{1}{n}.$$

This is not any term whatsoever of the series evaluated at z. The corrections required to make a convergent series also make a series that is not obviously $\mathbb{Z}$-periodic as a function of z, as it must be to represent the cotangent.

To show that the sum is meromorphic we shall use Theorem 3.4. To apply Theorem 3.4 here, let $\Omega = \mathbb{C} \setminus \mathbb{Z}$, a region in $\mathbb{C}$. Define

$$\varphi_n \colon \Omega \to \mathbb{C}, \qquad \varphi_n(z) = \frac{1}{z} + \sum_{j=1}^{n} \Big(\frac{1}{z-j} + \frac{1}{z+j}\Big), \qquad n \in \mathbb{N}.$$

This is the sequence of partial sums of

$$\varphi \colon \Omega \to \mathbb{C}, \qquad \varphi(z) = \frac{1}{z} + \sum_{j=1}^{\infty} \Big(\frac{1}{z-j} + \frac{1}{z+j}\Big).$$

Consider any $z \in \Omega$. For all $j > \sqrt{2}|z|$, the reverse triangle inequality gives

$$|z^2 - j^2| \geq j^2 - |z|^2 > j^2 - j^2/2 = j^2/2,$$

and so

$$\Big|\frac{1}{z^2-j^2}\Big| < \frac{2}{j^2}.$$

This shows that the partial sums

$$\varphi_n(z) = \frac{1}{z} + 2z \sum_{j=1}^{n} \frac{1}{z^2-j^2}$$

converge absolutely. Consequently, they converge to the limit function

$$\varphi(z) = \frac{1}{z} + 2z \sum_{j=1}^{\infty} \frac{1}{z^2-j^2}.$$

We need to show that the convergence is uniform on compact subsets of Ω. Let K be such a subset, and let $\varepsilon > 0$ be given. There is a uniform bound $b > 0$ on the absolute values $|z|$ for all $z \in K$. Also, there is a starting index $n_0 \in \mathbb{N}$ such that for any $n > n_0$,

$$\sum_{j=n+1}^{\infty} \frac{1}{j^2} < \varepsilon/(4b).$$

Consider any n such that $n > n_0$ and also $n > \sqrt{2}b$. For such n and for all $z \in K$,

$$|\varphi(z) - \varphi_n(z)| = \Big|2z \sum_{j=n+1}^{\infty} \frac{1}{z^2-j^2}\Big| \leq 2b \sum_{j=n+1}^{\infty} \Big|\frac{1}{z^2-j^2}\Big| \leq 2b \sum_{j=n+1}^{\infty} \frac{2}{j^2} < \varepsilon.$$

This shows that the convergence of $(\varphi_n : n \in \mathbb{N})$ on Ω is uniform on compact subsets.

By the result, the limit function can be differentiated termwise. Now that we no longer need the symbol n to index partial sums, we return to the more natural notation of using it as sum-index,

$$\varphi(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\Big(\frac{1}{z-n} + \frac{1}{z+n}\Big) = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2},$$

and

$$\varphi'(z) = -\sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2}.$$

The second series for φ shows that it is odd, and the series for φ' shows that it is even. The convergence of φ' is again absolute, and so φ' is $\mathbb{Z}$-periodic by a calculation that rearranges terms,

$$\varphi'(z+m) = -\sum_{n\in\mathbb{Z}}\frac{1}{(z+m-n)^2} = -\sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2}.$$

It follows that

$$\big(\varphi(z+1) - \varphi(z)\big)' = \varphi'(z+1) - \varphi'(z) = 0,$$

so that

$$\varphi(z+1) - \varphi(z) = c \qquad \text{for some constant } c.$$

To show that φ is $\mathbb{Z}$-periodic, we need to show that $c = 0$. But in particular,

$$c = \varphi(1/2) - \varphi(-1/2) = 2\varphi(1/2) \qquad \text{since } \varphi \text{ is odd,}$$

and so it suffices to show that $\varphi(1/2) = 0$. Inspect it,

$$\varphi(1/2) = 2 + \sum_{n=1}^{\infty}\frac{1}{1/4-n^2} = 2 - \sum_{n=1}^{\infty}\Big(\frac{1}{n-1/2} - \frac{1}{n+1/2}\Big).$$

The sum telescopes to 2, giving the desired result.

The argument so far shows that the function $\varphi(z) - 1/z$ is also analytic at $z = 0$. Therefore φ itself is meromorphic at 0, having a simple pole there with residue 1. By the $\mathbb{Z}$-periodicity, the same holds at each integer n. This matches the behavior of $\pi\cot\pi z$. Thus the difference $\pi\cot\pi z - \varphi(z)$ is entire. We want to show that it is the zero function.

The first step is to show that the difference is bounded, making it constant by Liouville's theorem. Since the difference is $\mathbb{Z}$-periodic in the x-direction, it suffices to show that is bounded as $|y| \to \infty$, and for this it suffices to show that each of $\pi\cot\pi z$ and $\varphi(z)$ is individually bounded as $|y| \to \infty$. Compute first that

$$\pi\cot\pi z = \pi i\frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} = \pi i\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1} = \pi i + \frac{2\pi i}{e^{2\pi iz}-1}.$$

Also $|e^{2\pi iz}| = e^{-2\pi y}$, so $\lim_{y\to+\infty}\pi\cot\pi z = -\pi i$ and $\lim_{y\to-\infty}\pi\cot\pi z = \pi i$. On the other hand, suppose now that $z = x+iy$ where $0 \le x < 1$ and $|y| > 1$. Then we have the inequalities $|y| \le |z| \le |y|+1$ and

$$|z^2 - n^2| = |x^2 - y^2 - n^2 + 2ixy| \ge y^2 + n^2 - x^2 \ge y^2 + n^2 - 1.$$

It follows that

$$|\varphi(z)| \le \frac{1}{|y|} + 2(|y|+1)\sum_{n=1}^{\infty}\frac{1}{y^2+n^2-1}.$$

Let $\eta = \lfloor |y| \rfloor$. Then

$$\sum_{n=1}^{\infty} \frac{1}{y^2 + n^2 - 1} = \sum_{m=0}^{\infty} \sum_{r=1}^{\eta} \frac{1}{y^2 + (m\eta + r)^2 - 1},$$

and for each $m \geq 0$,

$$\sum_{r=1}^{\eta} \frac{1}{y^2 + (m\eta + r)^2 - 1} \leq \frac{\eta}{\eta^2 + (m\eta)^2} \leq \frac{1}{\eta(1 + m^2)}.$$

This shows that

$$|\varphi(z)| \leq \frac{1}{|y|} + 2\frac{|y| + 1}{\lfloor |y| \rfloor} \sum_{m=0}^{\infty} \frac{1}{1 + m^2},$$

and so $\varphi(z)$ is bounded as $|y| \to \infty$ as well.

Thus $\pi \cot \pi z - \varphi(z)$ is constant. To see that the constant is 0, set $z = 1/2$. From before, $\varphi(1/2) = 0$. But also $\pi \cot \pi/2 = 0$, giving the result.

Example 3.30. As an application, we compare the power series expansions about $z = 0$ of the two now-known-to-be-equal functions

$$z\varphi(z) \qquad \text{and} \qquad \pi \cot \pi z.$$

For the first expansion, compute that for $|z| < 1$,

$$\begin{aligned} z\varphi(z) &= 1 + 2z^2 \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} \\ &= 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{1 - z^2/n^2} \\ &= 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{\infty} \left(\frac{z}{n}\right)^{2k} \\ &= 1 - 2 \sum_{k=0}^{\infty} z^{2k+2} \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \\ &= 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}. \end{aligned}$$

That is, $z\varphi(z)$ is a generating function for the Riemann zeta function $\zeta(2k)$ at positive even values. On the other hand, the second expansion is essentially a generating function for the Bernoulli numbers. Again for $|z| < 1$,

$$\begin{aligned} \pi z \cot \pi z &= \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} \\ &= \pi i z + \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i z)^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(2\pi i)^{2k} B_{2k}}{(2k)!} z^{2k}, \end{aligned}$$

since $B_{2k+1} \equiv 0$ for all $k \in \mathbb{N}$. Comparing these two expansions we get Euler's famous formula,

$$\zeta(2k) = -\frac{1}{2}\frac{(2\pi i)^{2k} B_{2k}}{(2k)!} \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.31}$$

In particular, this formula combines with the values $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ to give

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945},$$

and

$$\zeta(8) = \frac{\pi^8}{9450}, \qquad \zeta(10) = \frac{\pi^{10}}{93555}, \qquad \zeta(12) = \frac{691\pi^{12}}{638512875}.$$

Finally, we conclude that the formula (3.31) shows that $\zeta(2k)$ is transcendental for every $k \in \mathbb{N}$.

4. Simple properties of Gamma function and their applications

This section is based on [3] and [7].

4.1. Definitions of $\Gamma(z)$ and $B(z, w)$.

The following formula is valid:

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}. \tag{4.1}$$

This is an immediate consequence of the corresponding one-dimensional identity

$$\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi},$$

which is usually proved from its two-dimensional version by switching to polar coordinates:

$$I^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2} e^{-y^2}\, dx\, dy = 2\pi \int_0^\infty r e^{-r^2}\, dr = \pi.$$

Definition 4.2. For a complex number z with $\Re z > 0$ define

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

$\Gamma(z)$ is called the gamma function.

It follows from its definition that $\Gamma(z)$ is analytic on the right half-plane $\Re z > 0$. Two fundamental properties of the gamma function are that

$$\Gamma(z+1) = z\Gamma(z) \qquad \text{and} \qquad \Gamma(n) = (n-1)!,$$

where z is a complex number with positive real part and $n \in \mathbb{Z}^+$. Indeed, integration by parts yields

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt = \frac{t^z e^{-t}}{z}\bigg|_0^\infty + z^{-1}\int_0^\infty t^z e^{-t}\, dt = z^{-1}\Gamma(z+1).$$

Since $\Gamma(1) = 1$, the property $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}^+$ follows by induction. Another important fact is that

$$\Gamma(1/2) = \sqrt{\pi}.$$

This follows easily from the identity

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\, dt = 2\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}.$$

Next we define the beta function. Fix z and w complex numbers with positive real parts. We define

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt = \int_0^1 t^{w-1}(1-t)^{z-1}\,dt.$$

We have the following relationship between the gamma and the beta functions:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},$$

when z and w have positive real parts.

The proof of this fact is as follows:

$$\begin{aligned}
\Gamma(z+w)B(z,w) &= \Gamma(z+w)\int_0^1 t^{w-1}(1-t)^{z-1}\,dt \\
&= \Gamma(z+w)\int_0^\infty u^{w-1}(1+u)^{-z-w}\,du && \Big[t = u/(u+1)\Big] \\
&= \int_0^\infty\int_0^\infty u^{w-1}(1+u)^{-z-w}v^{z+w-1}e^{-v}\,dv\,du \\
&= \int_0^\infty\int_0^\infty u^{w-1}s^{z+w-1}e^{-s(u+1)}\,ds\,du && \Big[s = v/(u+1)\Big] \\
&= \int_0^\infty s^z e^{-s}\int_0^\infty (us)^{w-1}e^{-su}\,du\,ds \\
&= \int_0^\infty s^{z-1}e^{-s}\Gamma(w)\,ds \\
&= \Gamma(z)\Gamma(w).
\end{aligned}$$

4.2. **Volume of the Unit Ball and Surface of the Unit Sphere.** We denote by ν_n the volume of the unit ball in $\mathbb{R}^n$ and by ω_{n-1} the surface area of the unit sphere S^{n-1}. We have the following:

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

and

$$\nu_n = \omega_{n-1}/n = \frac{2\pi^{n/2}}{n\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

The easy proofs are based on the formula (4.1). We have

$$\pi^{n/2} = \int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \omega_{n-1}\int_0^\infty r^{n-1}e^{-r^2}\,dr,$$

by switching to polar coordinates. Now change variables $t = r^2$ to obtain that

$$\pi^{n/2} = \omega_{n-1}/2\int_0^\infty t^{n/2-1}e^{-t}\,dt = \Gamma(n/2)\omega_{n-1}/2.$$

This proves the formula for the surface area of the unit sphere in $\mathbb{R}^n$.

To compute ν_n, write again using polar coordinates

$$\nu_n = |B(0,1)| = \int_{|x|\le 1} dx = \int_{S^{n-1}}\int_0^1 r^{n-1}\,dr\,d\theta = \omega_{n-1}/n.$$

Here is another way to relate the volume to the surface area. Let $B(0,R)$ be the ball in $\mathbb{R}^n$ of radius $R > 0$ centered at the origin. Then the volume of the shell $B(0,R+h)\setminus B(0,R)$ divided by

h tends to the surface area of $B(0,R)$ as $h \to 0$. In other words, the derivative of the volume of $B(0,R)$ with respect to the radius R is equal to the surface area of $B(0,R)$. Since the volume of $B(0,R)$ is $\nu_n R^n$, it follows that the surface area of $B(0,R)$ is $n\nu_n R^{n-1}$. Taking $R=1$, we deduce $\omega_{n-1} = n\nu_n$.

4.3. **Computation of Integrals Using Gamma Functions.** Let $k_1, \dots, k_n$ be nonnegative even integers. The integral

$$\int_{\mathbb{R}^n} x_1^{k_1} \cdot \dots \cdot x_n^{k_n} e^{-|x|^2}\, dx_1 \dots dx_n = \prod_{j=1}^n \int_{\mathbb{R}} x_j^{k_j} e^{-x_j^2}\, dx_j = \prod_{j=1}^n \Gamma\Big(\frac{k_j+1}{2}\Big)$$

expressed in polar coordinates is equal to

$$\Big(\int_{S^{n-1}} \theta_1^{k_1} \cdot \dots \cdot \theta_n^{k_n}\, d\theta\Big) \int_0^\infty r^{k_1+\dots+k_n} r^{n-1} e^{-r^2}\, dr,$$

where $\theta = (\theta_1, \dots, \theta_n)$. This leads to the identity

$$\int_{S^{n-1}} \theta_1^{k_1} \cdot \dots \cdot \theta_n^{k_n}\, d\theta = 2\Gamma\Big(\frac{k_1+\dots+k_n+n}{2}\Big)^{-1} \prod_{j=1}^n \Gamma\Big(\frac{k_j+1}{2}\Big).$$

Another classical integral that can be computed using gamma functions is the following:

$$\int_0^{\pi/2} (\sin\varphi)^a (\cos\varphi)^b\, d\varphi = \frac{\Gamma\Big(\frac{a+1}{2}\Big)\Gamma\Big(\frac{b+1}{2}\Big)}{2\Gamma\Big(\frac{a+b+2}{2}\Big)},$$

whenever a and b are complex numbers with $\Re a > -1$ and $\Re b > -1$.

Indeed, change variables $u = \sin^2\varphi$; then $du = 2\sin\varphi\cos\varphi d\varphi$, and the preceding integral becomes

$$\frac{1}{2}\int_0^1 u^{(a-1)/2}(1-u)^{(b-1)/2}\, du = \frac{1}{2} B\Big(\frac{a+1}{2}, \frac{b+1}{2}\Big) = \frac{\Gamma\Big(\frac{a+1}{2}\Big)\Gamma\Big(\frac{b+1}{2}\Big)}{2\Gamma\Big(\frac{a+b+2}{2}\Big)}.$$

4.4. **Meromorphic Extensions of $B(z,w)$ and $\Gamma(z)$.** Using the identity $\Gamma(z+1) = z\Gamma(z)$, we can easily define a meromorphic extension of the gamma function on the whole complex plane starting from its known values on the right half-plane. We give an explicit description of the meromorphic extension of $\Gamma(z)$ on the whole plane. First write

$$\Gamma(z) = \int_0^1 t^{z-1} e^{-t}\, dt + \int_1^\infty t^{z-1} e^{-t}\, dt$$

and observe that the second integral is an analytic function of z for all $z \in \mathbb{C}$. Write the first integral as

$$\int_0^1 t^{z-1}\Big\{e^{-t} - \sum_{j=0}^N \frac{(-t)^j}{j!}\Big\}\, dt + \sum_{j=0}^N \frac{(-1)^j}{(z+j)\, j!}.$$

The last integral converges when $\Re z > -N-1$, since the expression inside the curly brackets is $O(t^{N+1})$ as $t \to 0$. It follows that the gamma function can be defined to be an analytic function on $\Re z > -N-1$ except at the points $z = -j$, $j = 0, 1, \dots, N$, at which it has simple poles with residues $\frac{(-1)^j}{j!}$. Since N was arbitrary, it follows that the gamma function has a meromorphic extension on the whole plane.

In view of the identity

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},$$

the definition of $B(z,w)$ can be extended to $\mathbb{C}\times\mathbb{C}$. It follows that $B(z,w)$ is a meromorphic function in each argument.

4.5. **Asymptotics of $\Gamma(x)$ as $x\to\infty$.** We now derive *Stirling's formula*:

$$\lim_{x\to\infty}\frac{\Gamma(x+1)}{(\frac{x}{e})^x\sqrt{2\pi x}}=1.$$

First change variables $t=x+sx\sqrt{2/x}$ to obtain

$$\Gamma(x+1)=\int_0^\infty e^{-t}t^x\,dt=\left(\frac{x}{e}\right)^x\sqrt{2x}\int_{-\sqrt{x/2}}^\infty\frac{\left(1+s\sqrt{2/x}\right)^x}{e^{2s\sqrt{x/2}}}\,ds.$$

Setting $y=\sqrt{x/2}$, we obtain

$$\frac{\Gamma(x+1)}{(\frac{x}{e})^x\sqrt{2x}}=\int_{\mathbb{R}}\left(\frac{\left(1+\frac{s}{y}\right)^y}{e^s}\right)^{2y}\chi_{(-y,\infty)}(s)\,ds.$$

To show that the last integral converges to $\sqrt{\pi}$ as $y\to\infty$, we need the following:

(1) The fact that

$$\lim_{y\to\infty}\left(\frac{\left(1+\frac{s}{y}\right)^y}{e^s}\right)^{2y}=e^{-s^2},$$

which follows easily by taking logarithms and applying L'Hôpital's rule twice.

(2) The estimate, valid for $y\geq 1$,

$$\left(\frac{\left(1+\frac{s}{y}\right)^y}{e^s}\right)^{2y}\leq\begin{cases}\frac{(1+s)^2}{e^s}, & s\geq 0,\\ e^{-s^2}, & -y<s<0,\end{cases}$$

which can be easily checked using calculus.

Using these facts, the Lebesgue dominated convergence theorem, the trivial fact that $\chi_{(-y,\infty)}(s)\to 1$ as $y\to\infty$, and the identity (4.1), we obtain that

$$\lim_{x\to\infty}\frac{\Gamma(x+1)}{(\frac{x}{e})^x\sqrt{2x}}=\lim_{y\to\infty}\int_{\mathbb{R}}\left(\frac{\left(1+\frac{s}{y}\right)^y}{e^s}\right)^{2y}\chi_{(-y,\infty)}(s)\,ds=\int_{\mathbb{R}}e^{-s^2}\,ds=\sqrt{\pi}.$$

As a consequence of Stirling's formula, for any $t>0$, we obtain

$$\lim_{x\to\infty}\frac{\Gamma(x)}{\Gamma(x+t)}=0.$$

4.6. **Stirling's formula: a more precise bounds.** We shall prove Stirling's formula by showing the following result.

Theorem 4.3. *For $n=1,2,\ldots$, we have*

$$n!=\sqrt{2\pi}n^{n+1/2}e^{-n}e^{r_n}\tag{4.4}$$

where r_n satisfies the double inequality

$$\frac{1}{12n+1}<r_n<\frac{1}{12n}.\tag{4.5}$$

The usual textbook proofs replace the first inequality in (4.5) by the weaker inequality

$$r_n > 0$$

or

$$r_n > \frac{1}{12n+6}.$$

Proof. Let

$$S_n = \log(n!) = \sum_{p=1}^{n-1} \log(p+1)$$

and write

$$\log(p+1) = A_p + b_p - \varepsilon_p, \tag{4.6}$$

where

$$\begin{aligned} A_p &= \int_p^{p+1} \log x \, dx, \\ b_p &= [\log(p+1) - \log p]/2, \\ \varepsilon_p &= \int_p^{p+1} \log x \, dx - [\log(p+1) + \log p]/2. \end{aligned}$$

The partition (4.6) of $\log(p+1)$, regarded as the area of a rectangle with base $(p, p+1)$ and height $\log(p+1)$, into a curvilinear area, a triangle, and a small sliver is suggested by the geometry of the curve $y = \log x$. Then

$$S_n = \sum_{p=1}^{n-1}(A_p + b_p - \varepsilon_p) = \int_1^n \log x \, dx + \frac{1}{2}\log n - \sum_{p=1}^{n-1} \varepsilon_p.$$

Since $\int \log x \, dx = x \log x - x$ we can write

$$S_n = (n + 1/2)\log n - n + 1 - \sum_{p=1}^{n-1} \varepsilon_p, \tag{4.7}$$

where

$$\varepsilon_p = \frac{2p+1}{2} \log\left(\frac{p+1}{p}\right) - 1.$$

Using the well known series

$$\log\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

valid for $|x| < 1$, and setting $x = (2p+1)^{-1}$, so that $(1+x)/(1-x) = (p+1)/p$, we find that

$$\varepsilon_p = \sum_{k=0}^{\infty} \frac{1}{(2k+3)(2p+1)^{2k+2}}. \tag{4.8}$$

We can therefore bound ε_p above and below:

$$\varepsilon_p < \frac{1}{3(2p+1)^2} \sum_{k=0}^{\infty} \frac{1}{(2p+1)^{2k}} = \frac{1}{12}\Big(\frac{1}{p} - \frac{1}{p+1}\Big), \tag{4.9}$$

$$\varepsilon_p > \frac{1}{3(2p+1)^2} \sum_{k=0}^{\infty} \frac{1}{[3(2p+1)^2]^k} = \frac{1}{3(2p+1)^2} \frac{1}{1 - \frac{1}{3(2p+1)^2}} > \frac{1}{12}\Big(\frac{1}{p+1/12} - \frac{1}{p+1+1/12}\Big). \tag{4.10}$$

Now define

$$B = \sum_{p=1}^{\infty} \varepsilon_p, \qquad r_n = \sum_{p=n}^{\infty} \varepsilon_p, \tag{4.11}$$

where from (4.9) and (4.10) we have

$$1/13 < B < 1/12. \tag{4.12}$$

Then we can write (4.7) in the form

$$S_n = (n + 1/2)\log n - n + 1 - B + r_n,$$

or, setting $C = e^{1-B}$, as

$$n! = C n^{n+1/2} e^{-n} e^{r_n},$$

where r_n is defined by (4.11), ε_p by (4.8), and from (4.9) and (4.10) we have

$$1/(12n+1) < r_n < 1/(12n).$$

The constant C, known from (4.12) to lie between $e^{11/12}$ and $e^{12/13}$, may be shown by one of the usual methods to have the value $\sqrt{2\pi}$. This completes the proof. □

4.7. **Euler's Limit Formula for the Gamma Function.** For n a positive integer and $\Re z > 0$ we consider the functions

$$\Gamma_n(z) = \int_0^n \Big(1 - \frac{t}{n}\Big)^n t^{z-1}\, dt.$$

We show that

$$\Gamma_n(z) = \frac{n! n^z}{z(z+1)\cdot \ldots \cdot (z+n)}$$

and we obtain *Euler's limit formula for the gamma function*

$$\lim_{n\to\infty} \Gamma_n(z) = \Gamma(z).$$

We write $\Gamma(z) - \Gamma_n(z) = I_1(z) + I_2(z) + I_3(z)$, where

$$I_1(z) = \int_n^\infty e^{-t} t^{z-1}\, dt,$$

$$I_2(z) = \int_{n/2}^{n} \Big(e^{-t} - \Big(1 - \frac{t}{n}\Big)^n\Big) t^{z-1}\, dt,$$

$$I_3(z) = \int_0^{n/2} \Big(e^{-t} - \Big(1 - \frac{t}{n}\Big)^n\Big) t^{z-1}\, dt.$$

Obviously $I_1(z)$ tends to zero as $n \to \infty$. For I_2 and I_3 we have that $0 \le t < n$, and by the Taylor expansion of the logarithm we obtain

$$\log\left(1-\frac{t}{n}\right)^n = n\log\left(1-\frac{t}{n}\right) = -t - L,$$

where

$$L = \frac{t^2}{n}\sum_{k=0}^{\infty}\frac{t^k}{(k+2)n^k}.$$

It follows that

$$0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n = e^{-t} - e^{-L}e^{-t} \le e^{-t},$$

and thus $I_2(z)$ tends to zero as $n \to \infty$. For I_3 we have $t/n \le 1/2$, which implies that

$$L \le \frac{t^2}{n}\sum_{k=0}^{\infty}\frac{1}{(k+1)2^{k-1}} = \frac{t^2}{n}c.$$

Consequently, for $t/n \le 1/2$ we have

$$0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n = e^{-t}(1-e^{-L}) \le e^{-t}L \le e^{-t}\frac{ct^2}{n}.$$

Plugging this estimate into I_3, we deduce that

$$|I_3(z)| \le \frac{c}{n}\Gamma(\Re z + 2),$$

which certainly tends to zero as $n \to \infty$.

Next, n integrations by parts give

$$\Gamma_n(z) = \left(\prod_{k=0}^{n-1}\frac{n-k}{n(z+k)}\right)\int_0^n t^{z+n-1}\,dt = \frac{n!\,n^z}{z(z+1)\cdot\ldots\cdot(z+n)}.$$

This can be written as

$$1 = \Gamma_n(z)z\exp\left\{z\left(\sum_{k=1}^{n}k^{-1} - \log n\right)\right\}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-z/k}.$$

Taking limits as $n \to \infty$, we obtain an *infinite product form of Euler's limit formula*,

$$1 = \Gamma(z)ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k},$$

where $\Re z > 0$ and γ is Euler's constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}k^{-1} - \log n\right).$$

The infinite product converges uniformly on compact subsets of the complex plane that excludes $z = 0, -1, -2, \ldots$, and thus it represents a holomorphic function in this domain. This holomorphic function multiplied by $\Gamma(z)ze^{\gamma z}$ is equal to 1 on $\Re z > 0$ and by analytic continuation it must be equal to 1 on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$. But $\Gamma(z)$ has simple poles, while the infinite product vanishes to order one, at the nonpositive integers. We conclude that Euler's limit formula holds for all complex numbers z; consequently, $\Gamma(z)$ has no zeros and $\Gamma(z)^{-1}$ is entire.

An immediate consequence of Euler's limit formula is the identity

$$\frac{1}{|\Gamma(x+iy)|^2} = \frac{1}{|\Gamma(x)|^2}\prod_{k=0}^{\infty}\Big(1+\frac{y^2}{(k+x)^2}\Big),$$

which holds for x and y real with $x \notin \{0,-1,\ldots\}$. As a consequence we have that

$$|\Gamma(x+iy)| \le |\Gamma(x)|$$

and also that

$$\frac{1}{|\Gamma(x+iy)|} \le \frac{1}{|\Gamma(x)|}e^{C(x)|y|^2},$$

where

$$C(x) = \frac{1}{2}\sum_{k=0}^{\infty}(k+x)^{-2},$$

whenever $y \in \mathbb{R}$ and $x \in \mathbb{R}\setminus\{0,-1,\ldots\}$. Before we find a similar estimate for $x \in \{0,-1,\ldots\}$ we provide a simpler expression for this estimate when $x > 0$.

When $x > 0$ we have

$$C(x) \le \frac{1}{2x^2}+\frac{1}{2}\sum_{k=1}^{\infty}(k+x)^{-2} \le \frac{1}{2x^2}+\frac{1}{2}\int_0^{\infty}\frac{dt}{(t+x)^2} = \frac{1}{2x^2}+\frac{1}{2x}\int_1^{\infty}\frac{dt}{t^2} = \frac{1}{2x^2}+\frac{1}{2x}.$$

Thus we conclude that when $x > 0$ and $y \in \mathbb{R}$ we have

$$\frac{1}{|\Gamma(x+iy)|} \le \frac{1}{|\Gamma(x)|}e^{\max\{x^{-2},x^{-1}\}|y|^2}.$$

When $x = 0$ we write $\Gamma(iy)iy = \Gamma(1+iy)$ and use the preceding inequality to obtain

$$\frac{1}{|\Gamma(iy)|} \le \frac{|iy|}{|\Gamma(1)|}e^{|y|^2} = |y|e^{|y|^2}$$

and more generally for $x = -N \in \{-1,-2,\ldots\}$ and $y \in \mathbb{R}$ we obtain by induction

$$\frac{1}{|\Gamma(-N+iy)|} \le |iy||1+iy|\cdot\ldots\cdot|N+iy|e^{|y|^2}.$$

4.8. **Reflection and Duplication Formulas for the Gamma Function.** The *reflection formula* relates the values of the gamma function of a complex number z and its reflection about the point $1/2$ in the following way:

$$\frac{\sin(\pi z)}{\pi} = \frac{1}{\Gamma(z)}\frac{1}{\Gamma(1-z)}.$$

The *duplication formula* relates the entire functions $\Gamma(2z)^{-1}$ and $\Gamma(z)^{-1}$ as follows:

$$\frac{1}{\Gamma(z)\Gamma(z+1/2)} = \frac{\pi^{-1/2}2^{2z-1}}{\Gamma(2z)}.$$

Both of these could be proved using Euler's limit formula. The reflection formula also uses the identity

$$\prod_{k=1}^{\infty}\Big(1-\frac{z^2}{k^2}\Big) = \frac{\sin(\pi z)}{\pi z},$$

while the duplication formula makes use of the fact that

$$\lim_{n\to\infty} \frac{(n!)^2 2^{2n+1}}{(2n)!n^{1/2}} = 2\sqrt{\pi}.$$

These and other facts related to the gamma function can be found in Olver [5].

5. Laplace's method and Stirling's formula again

This section is based on [9].

Proposition 5.1. *Suppose a and m are fixed, with $a > 0$ and $m > -1$. Then as $s \to \infty$*

$$\int_0^a e^{-sx} x^m \, dx = s^{-m-1}\Gamma(m+1) + O(e^{-cs}), \tag{5.2}$$

for some positive c.

Proof. The fact that $m > -1$ guarantees that the integral on the left-hand side exists. Then, we write

$$\int_0^a e^{-sx} x^m \, dx = \int_0^\infty e^{-sx} x^m \, dx - \int_a^\infty e^{-sx} x^m \, dx.$$

The first integral on the right-hand side can be seen to equal $s^{-m-1}\Gamma(m+1)$, if we make the change of variables $x \mapsto x/s$. For the second integral we note that

$$\int_a^\infty e^{-sx} x^m \, dx = e^{-cs} \int_a^\infty e^{-s(x-c)} x^m \, dx = O(e^{-cs}), \tag{5.3}$$

as long as $c < a$, and so the proposition is proved. □

For later purposes it is interesting to point out that under certain restricted circumstances, the gist of the conclusion in Proposition 5.1 extends to the complex half-plane $\Re s \geq 0$.

Proposition 5.4. *Suppose a and m are fixed, with $a > 0$ and $-1 < m < 0$. Then as $|s| \to \infty$ with $\Re s \geq 0$,*

$$\int_0^a e^{-sx} x^m \, dx = s^{-m-1}\Gamma(m+1) + O(1/|s|).$$

(Here s^{-m-1} is the branch of that function that is positive for $s > 0$.)

Proof. We begin by showing that when $\Re s \geq 0$, $s \neq 0$,

$$\int_0^\infty e^{-sx} x^m \, dx = \lim_{N\to\infty} \int_0^N e^{-sx} x^m \, dx$$

exists and equals $s^{-m-1}\Gamma(m+1)$. If N is large, we first write

$$\int_0^N e^{-sx} x^m \, dx = \int_0^a e^{-sx} x^m \, dx + \int_a^N e^{-sx} x^m \, dx.$$

Since $m > -1$, the first integral on the right-hand side defines an analytic function everywhere. For the second integral, we note that $-s^{-1}\partial_x(e^{-sx}) = e^{-sx}$, so an integration by parts gives

$$\int_a^N e^{-sx} x^m \, dx = \frac{m}{s} \int_a^N e^{-sx} x^{m-1} \, dx - \frac{e^{-sx}}{s} x^m \Big|_a^N. \tag{5.5}$$

This identity, together with the convergence of the integral $\int_a^\infty x^{m-1} \, dx$, shows that $\int_a^\infty e^{-sx} x^m \, dx$ defines an analytic function on $\Re s > 0$ that is continuous on $\Re s \geq 0$, $s \neq 0$. Thus $\int_0^\infty e^{-sx} x^m \, dx$ is analytic on the half-plane $\Re s > 0$ and continuous on $\Re s \geq 0$, $s \neq 0$. Since it equals $s^{-m-1}\Gamma(m+1)$ when s is positive, we deduce that $\int_0^\infty e^{-sx}x^m\,dx = s^{-m-1}\Gamma(m+1)$ when $\Re s \geq 0$, $s \neq 0$.

However, we now have

$$\int_0^a e^{-sx}x^m\,dx = \int_0^\infty e^{-sx}x^m\,dx - \int_a^\infty e^{-sx}x^m\,dx.$$

It is clear from (5.5), and from the fact that $m < 0$, that if we let $N \to \infty$, then $\int_a^\infty e^{-sx}x^m\,dx = O(1/|s|)$. The proposition is therefore proved. □

Note. If one wants to obtain a better error term in Proposition 5.4, or for that matter extend the range of m, then one needs to mitigate the effect of the contribution of the end-point $x = a$. This can be done by introducing suitable smooth cut-offs.

5.1. **Laplace's method.** We have already mentioned that when Φ is real-valued, the main contribution to $\int_a^b e^{-s\Phi(x)}\,dx$ as $s \to \infty$ comes from the point where Φ takes its minimum value. A situation where this minimum is attained at one of the end-points, a or b, was considered in Proposition 5.1. We now turn to the important case when the minimum is achieved in the interior of $[a, b]$.

Consider

$$\int_a^b e^{-s\Phi(x)}\psi(x)\,dx$$

where the *phase* Φ is real-valued, and both it and the *amplitude* ψ are assumed for simplicity to be indefinitely differentiable. Our hypothesis regarding the minimum of Φ is that there is an $x_0 \in (a, b)$ so that $\Phi'(x_0) = 0$, but $\Phi''(x_0) > 0$ throughout $[a, b]$.

Proposition 5.6. *Under the above assumptions, with $s > 0$ and $s \to \infty$,*

$$\int_a^b e^{-s\Phi(x)}\psi(x)\,dx = e^{-s\Phi(x_0)}\Big[\frac{A}{s^{1/2}} + O(s^{-1})\Big], \tag{5.7}$$

where

$$A = \sqrt{2\pi}\frac{\psi(x_0)}{(\Phi''(x_0))^{1/2}}.$$

Proof. By replacing $\Phi(x)$ by $\Phi(x) - \Phi(x_0)$ we may assume that $\Phi(x_0) = 0$. Since $\Phi'(x_0)$, we note that

$$\frac{\Phi(x)}{(x-x_0)^2} = \frac{\Phi''(x_0)}{2}\varphi(x),$$

where φ is smooth, and $\varphi(x) = 1 + O(x - x_0)$ as $x \to x_0$. We can therefore make the smooth change of variables $x \mapsto y = (x - x_0)(\varphi(x))^{1/2}$ in a small neighborhood of $x = x_0$, and observe that $dy/dx|_{x_0} = 1$, and thus $dx/dy = 1 + O(y)$ as $y \to 0$. Moreover, we have $\psi(x) = \tilde{\psi}(y)$ with $\tilde{\psi}(y) = \psi(x_0) + O(y)$ as $y \to 0$. Hence if $[a', b']$ is a sufficiently small interval containing x_0 in its interior, by making the indicated change of variables we obtain

$$\int_{a'}^{b'} e^{-s\Phi(x)}\psi(x)\,dx = \psi(x_0)\int_\alpha^\beta e^{-s\Phi''(x_0)y^2/2}\,dy + O\Big(\int_\alpha^\beta e^{-s\Phi''(x_0)y^2/2}|y|\,dy\Big), \tag{5.8}$$

where $\alpha < 0 < \beta$. We now make the further change of variables $y^2 = X$, $dy = \frac{1}{2}X^{-1/2}dX$, and we see by (5.2) that the first integral on the right-hand side in (5.8) is

$$\int_0^{a_0} e^{-s\Phi''(x_0)X/2}X^{-1/2}\,dX + O(e^{-\delta s}) = s^{-1/2}\Big(\frac{2\pi}{\Phi''(x_0)}\Big)^{1/2} + O(e^{-\delta s}),$$

for some $\delta > 0$. By the same argument, the second integral is $O(1/s)$. What remains are the integrals of $e^{-s\Phi(x)}\psi(x)$ over $[a, a']$ and $[b', b]$; but these integrals decay exponentially as $s \to \infty$, since $\Phi(x) \geq c > 0$ in these two sub-intervals. Altogether, this establishes (5.7) and the proposition. $\square$

It is important to realize that the asymptotic relation (5.7) extends to all complex s with $\Re s \geq 0$. The proof, however, requires a somewhat different argument: here we must take into account the oscillations of $e^{-s\Phi(x)}$ when $|s|$ is large but $\Re s$ is small, and this is achieved by a simple integration by parts.

Proposition 5.9. *With the same assumptions on Φ and ψ, the relation (5.7) continues to hold if $|s| \to \infty$ with $\Re s \geq 0$.*

Proof. We proceed as before to the equation (5.8), and obtain the appropriate asymptotic for the first term, by virtue of Proposition 5.4, with $m = -1/2$. To deal with the rest we start with an observation. If Ψ and ψ are given on an interval $[\bar{a}, \bar{b}]$, are indefinitely differentiable, and $\Psi(x) \geq 0$, while $|\Psi'(x)| \geq c > 0$, then if $\Re s \geq 0$,

$$\int_{\bar{a}}^{\bar{b}} e^{-s\Psi(x)}\psi(x)\, dx = O(|s|^{-1}) \qquad \text{as } |s| \to \infty. \tag{5.10}$$

Indeed, the integral equals

$$-s^{-1} \int_{\bar{a}}^{\bar{b}} \partial_x \big(e^{-s\Psi(x)}\big) \frac{\psi(x)}{\Psi'(x)}\, dx,$$

which by integration by parts gives

$$s^{-1} \int_{\bar{a}}^{\bar{b}} e^{-s\Psi(x)} \partial_x \Big(\frac{\psi(x)}{\Psi'(x)}\Big)\, dx - s^{-1}\Big[e^{-s\Psi(x)}\frac{\psi(x)}{\Psi'(x)}\Big]_{\bar{a}}^{\bar{b}}.$$

The assertion (5.10) follows immediately since $|e^{-s\Psi(x)}| \leq 1$, when $\Re s \geq 0$. This allows us to deal with the integrals of $e^{-s\Phi(x)}\psi(x)$ in the complementary intervals $[a, a']$ and $[b', b]$, because in each, $|\Phi'(x)| \geq c > 0$, since $\Phi'(x_0) = 0$ and $\Phi''(x) \geq c_1 > 0$.

Finally, for the second term on the right-hand side of (5.8) we observe that it is actually of the form

$$\int_{\alpha}^{\beta} e^{-s\Phi''(x_0)y^2/2} y\eta(y)\, dy,$$

where $\eta(y)$ is differentiable. Then we can again estimate this term by integration by parts, once we write it as

$$-\frac{1}{s\Phi''(x_0)} \int_{\alpha}^{\beta} \partial_y \Big(e^{-s\Phi''(x_0)y^2/2}\Big)\eta(y)\, dy,$$

obtaining the bound $O(|s|^{-1})$. $\square$

The special case of Proposition 5.9 when s is purely imaginary, $s = it$, $t \to \pm\infty$, is often treated separately; the argument in this situation is usually referred to as the method of *stationary phase.* The points x_0 for which $\Phi'(x_0) = 0$ are called the *critical points.*

5.2. **Stirling's formula in the complex domain.** Our application will be to the asymptotic behavior of the gamma function Γ, given by Stirling's formula. This formula will be valid in any sector of the complex plane that omits the negative real axis. For any $\delta > 0$ we set $S_\delta = \{s : |\arg s| \leq \pi - \delta\}$, and denote by $\log s$ the principal branch of the logarithm that is given in the plane slit along the negative real axis.

Theorem 5.11. *If $|s| \to \infty$ with $s \in S_\delta$, then*

$$\Gamma(s) = e^{s \log s} e^{-s} \frac{\sqrt{2\pi}}{s^{1/2}} \Big(1 + O(|s|^{-1/2})\Big). \tag{5.12}$$

Remark 5.13. With a little extra effort one can improve the error term to $O(1/|s|)$, and in fact obtain a complete asymptotic expansion in powers of $1/s$. Also, we note that (5.12) implies $\Gamma(s) \sim \sqrt{2\pi} s^{s-1/2} e^{-s}$, which is how Stirling's formula is often stated.

Proof. To prove the theorem we first establish (5.12) in the right half-plane. We shall show that the formula holds whenever $\Re s > 0$, and in addition that the error term is uniform on the closed half-plane, once we omit a neighborhood of the origin (say $|s| < 1$). To see this, start with $s > 0$, and write

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x} = \int_0^\infty e^{-x + s \log x} \frac{dx}{x}.$$

Upon making the change of variables $x \mapsto sx$, the above equals

$$\int_0^\infty e^{-sx + s \log sx} \frac{dx}{x} = e^{s \log s} e^{-s} \int_0^\infty e^{-s\Phi(x)} \frac{dx}{x},$$

where $\Phi(x) = x - 1 - \log x$. By analytic continuation this identity continues to hold, and we have when $\Re s > 0$,

$$\Gamma(s) = e^{s \log s} e^{-s} I(s)$$

with

$$I(s) = \int_0^\infty e^{-s\Phi(x)} \frac{dx}{x}.$$

It now suffices to see that

$$I(s) = \frac{\sqrt{2\pi}}{s^{1/2}} + O(|s|^{-1}) \qquad \text{for } \Re s > 0. \tag{5.14}$$

Observe first that $\Phi(1) = \Phi'(1) = 0$, $\Phi''(x) = x^{-2} > 0$ whenever $0 < x < \infty$, and $\Phi''(1) = 1$. Thus Φ is convex, attains its minimum at $x = 1$, and is positive.

We apply the complex version of the Laplace method, Proposition 5.9, in this situation. Here the critical point is $x_0 = 1$ and $\psi(x) = 1/x$. We choose for convenience the interval $[a, b]$ to be $[1/2, 2]$. Then for $\int_a^b e^{-s\Phi(x)} \psi(x)\, dx$ we get the asymptotic (5.14). It remains to bound the error terms, those corresponding to integration over $[0, 1/2]$, and $[2, \infty)$. Here the device of integration by parts, which has served us so well, can be applied again. Indeed, since $\Phi'(x) = 1 - 1/x$, we have

$$\begin{aligned}\int_\varepsilon^{1/2} e^{-s\Phi(x)} \frac{dx}{x} &= s^{-1} \int_\varepsilon^{1/2} \partial_x \Big(e^{-s\Phi(x)}\Big) \frac{dx}{\Phi'(x) x} \\ &= -s^{-1} \left[\frac{e^{-s\Phi(x)}}{x-1}\right]_\varepsilon^{1/2} - s^{-1} \int_\varepsilon^{1/2} e^{-s\Phi(x)} \frac{dx}{(x-1)^2}.\end{aligned}$$

Noting that $\Phi(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, and $|e^{-s\Phi(x)}| \leq 1$, we find in the limit that

$$\int_0^{1/2} e^{-s\Phi(x)} \frac{dx}{x} = 2s^{-1} e^{-s\Phi(1/2)} - s^{-1} \int_0^{1/2} e^{-s\Phi(x)} \frac{dx}{(x-1)^2}.$$

Thus the left-hand side is $O(|s|^{-1})$ in the half-plane $\Re s \geq 0$.

The integral $\int_2^\infty e^{-s\Phi(x)} \frac{dx}{x}$ is treated analogously, once we note that $\int_2^\infty (x-1)^{-2}\, dx$ converges.

Since these estimates are uniform, (5.14) and thus (5.12) are proved for $\Re s \geq 0$, $|s| \to \infty$.

To pass from $\Re s \geq 0$ to $\Re s \leq 0$, $s \in S_\delta$, we record the following fact about the principal branch of $\log s$: whenever $\Re s \geq 0$, $s = \sigma + it$, $t \neq 0$, then

$$\log(-s) = \begin{cases} \log s - i\pi, & t > 0, \\ \log s + i\pi, & t < 0. \end{cases}$$

Hence if $G(s) = e^{s \log s} e^{-s}$, $\Re s \geq 0$, $t \neq 0$, then

$$G(-s)^{-1} = \begin{cases} e^{s \log s} e^{-s} e^{-si\pi}, & t > 0, \\ e^{s \log s} e^{-s} e^{si\pi}, & t < 0. \end{cases} \tag{5.15}$$

Next,

$$\Gamma(s)\Gamma(-s) = \frac{\pi}{-s \sin \pi s}, \tag{5.16}$$

which follows from the fact that $\Gamma(s)\Gamma(1-s) = \pi / \sin \pi s$, and $\Gamma(1-s) = -s\Gamma(-s)$. The combination of (5.15) and (5.16), together with the fact that for large s, $\big(1+O(|s|^{-1/2})\big)^{-1} = 1+O(|s|^{-1/2})$, allows us then to extend (5.12) to the whole sector S_δ, thereby completing the proof of the theorem. □

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