Vector Bundles and Projective Modules

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Serre-Swan Correspondence

Serre-Swan Correspondence

If X is a compact Hausdorff space the category of complex vector bundles over X is equivalent to the category of finitely generated projective $C(X)$-modules.

$$\begin{gathered}\{vector\ bundles\ over\ X\} \\ \simeq \\ \{finitely\ generated\ projective\ C(X) - modules\}\end{gathered} \tag{1}$$

The functor which establishes the equivalence will be called the cross section functor Γ.

Remark

The Serre-Swan Correspondence will allows us to define a noncommutative version of a vector bundle!

In A Nutshell...

$$\text{trivial vector bundles } X \times \mathbb{C}^n \longleftrightarrow \text{free } C(X) - \text{module } [C(X)]^n \qquad (2)$$

- Over a compact Hausdorff space every vector bundle E can be trivialized, i.e, there exists E' such that

$$E \oplus E' \simeq \text{trivial} \qquad (3)$$

- A projective module P is one which can be trivialized in the sense that there exists P' such that

$$P \oplus P' \simeq \text{free} \qquad (4)$$

Category of Vector Bundles

- **Objects**: vector bundles E over X, i.e, E is a topological space with a map $p : E \longrightarrow X$ such that $p^{-1}(x)$ has the structure of a vector space and which is locally trivial, for each $x \in X$ there exists a neighborhood U_x of x such that $p^{-1}(U_x)$ is homeomorphic to the product bundle $U_x \times \mathbb{C}^n$.
- **Arrows**: given vector bundles E, E' over X, a morphism is a map $f : E \longrightarrow E'$ such that it is fiber wise linear and the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow_{p} \quad \swarrow_{p'} & \\ & X & \end{array} \tag{5}$$

i.e, $f(E_x) \subseteq E'_x$.

Category of Finitely Generated Projective Modules

- **Objects**: finitely generated projective $C(X)$ modules, i.e, P is a module over $C(X)$ for which there exists a $C(X)$ module Q such that $P \oplus Q$ is a free $C(X)$ module of finite rank.
- **Arrows**: if P, P' are finitely generated projective $C(X)$ modules, an arrow $f : P \longrightarrow P'$ is simply a $C(X)$ module homomorphism

Equivalence of Categories

We will construct a "map" (functor) Γ such that

1. If E is a vector bundle over X, then ΓE is a finitely generated projective $C(X)$ module.
2. If $f : E \longrightarrow E'$ is a morphism between vector modules, then $\Gamma f : \Gamma E \longrightarrow \Gamma E'$ is a module homomorphism
3. The application $f \longrightarrow \Gamma f$ is injective
4. For every $g : \Gamma E \longrightarrow \Gamma E'$ there exists $f : E \longrightarrow E'$ such that $g = \Gamma f$ (surjectivity)
5. For every finitely generated projective module P there exists a vector bundle E such that ΓE and P are isomorphic

The Cross Section Functor Γ

- Given a vector bundle over E, we define ΓE to be the global cross-sections of E

$$\Gamma E \equiv \{s : X \longrightarrow E \quad p \circ s = 1_X\} \tag{6}$$

Clearly ΓE is a $C(X)$ module by

$$\begin{cases} (h \cdot s)(x) = h(x)\, s(x) \\ (s_1 + s_2)(x) = s_1(x) + s_2(x) \end{cases} \tag{7}$$

where $h : X \longrightarrow \mathbb{C}$.

- Need to show that ΓE is a finitely generated projective module

ΓE is a finitely generated projective module

We will prove the following:

1. We can take $Q = \Gamma(E')$ where $E \oplus E'$ is a trivial vector bundle. $E \oplus E'$ is the Whitney sum:

$$E \oplus E' = \{(e, e') : e \in E, e' \in E', \quad p(e) = p'(e')\} \quad (8)$$

2. We can see

$$\Gamma(E) \oplus \Gamma(E') \simeq \Gamma(E \oplus E') \simeq \Gamma\left(X \times \mathbb{C}^N\right) \simeq (C(X))^N \quad (9)$$

where if $s'' \in \Gamma(E \oplus E')$ we define $s = p \circ s'' \in \Gamma(E)$ and $s' = p' \circ s'' \in \Gamma(E')$. This will show that ΓE is finitely generated projective module.

Trivialization Property

- We can find open sets $U_1, \cdots, U_m$ and $V_1, \cdots, V_m$ where E is trivial on each V_i, $U_i \subseteq V_i$ and there exists $\xi_i : X \longrightarrow [0,1]$ with $\mathrm{supp}\xi_i \subseteq V_i$, $\xi_i = 1$ on U_i and the U_i are an open cover for X.
- We have $\varphi_i : p^{-1}(V_i) \longrightarrow (V_i, \mathbb{C}^n)$ and can write $\varphi_i(v) = (p(v), \psi(v))$ where $\psi_i : p^{-1}(V_i) \longrightarrow \mathbb{C}^n$.
- Define $\Psi : E \longrightarrow (\mathbb{C}^n)^m$ and $\Phi : E \longrightarrow X \times (\mathbb{C}^n)^m$ by

$$\begin{aligned} \Psi(v) &= (\xi_i(p(v))\psi_i(v))_{1 \leq i \leq m} \\ \Phi(v) &= (p(x), \Psi(v)) \end{aligned} \tag{10}$$

Since $\Psi_{|E_x}$ is a linear monomorphism for each x we can identify E with $\Phi(E)$ which is a subbundle of the trivial bundle $X \times (\mathbb{C}^n)^m$.

Definition of Γf

- Suppose that $f : E \longrightarrow E'$ is a vector bundle morphism.
- Define $\Gamma f : \Gamma E \longrightarrow \Gamma E'$ by

$$(\Gamma f)(s)(x) = (f \circ s)(x) \tag{11}$$

where $s : X \longrightarrow E$ is a section of E. Observe that $(\Gamma f)(s)$ is a section because

$$p' \circ (\Gamma f(s)) = p' \circ (f \circ s) = (p' \circ f) \circ s = p \circ s = 1_X \tag{12}$$

- It is a $C(X)$ module homomorphism because of the fiber wise linearity of f

$$\begin{aligned}(\Gamma f)(h \cdot s + s')(x) &= (f \circ (h \cdot s + s'))(x) = f(h(x)s(x) + s'(x)) \\ &= (\Gamma f)(h \cdot s)(x) + (\Gamma f)(s')(x)\end{aligned} \tag{13}$$

Injectivity of $f \longrightarrow \Gamma f$

- Suppose that $\Gamma f = \Gamma g$ for $f, g : E \longrightarrow E'$ morphisms.
- If $e \in E$ let $x = p(e)$. If we take U_x a trivializing open neighborhood for x then there exists a local section $s_{U_x} : U_x \longrightarrow E$ such that $s_{U_x}(x) = e$. By Urysohn's Lemma we can find $\varphi : X \longrightarrow [0,1]$ such that $\varphi(x) = 1$ and $\mathrm{supp}\varphi \subseteq U_x$. Then $\varphi \cdot s_{U_x} : X \longrightarrow E$ is a global section, therefore

$$\begin{aligned} f(e) = f(\varphi \cdot s_{U_x}(x)) = (\Gamma f)(\varphi \cdot s_{U_x})(x) \\ = (\Gamma g)(\varphi \cdot s_{U_x})(x) = g(\varphi \cdot s_{U_x}(x)) = g(e) \end{aligned} \tag{14}$$

which shows that $f = g$.

Surjectivity of Γ

- Suppose that $g : \Gamma(E) \longrightarrow \Gamma(E')$ is a $C(X)$ homomorphism.
- We want $f : E \longrightarrow E'$ such that $\Gamma f = g$. We will define it fiberwise because if $e \in E$ then e belongs to some fiber.
- For $x \in X$ consider the evaluation maps

$$\begin{aligned} \varepsilon_x : C(X) \longrightarrow \mathbb{C} \qquad & h \longrightarrow h(x) \\ \varepsilon_x^E : \Gamma(E) \longrightarrow E_x \qquad & s \longrightarrow s(x) \end{aligned} \tag{15}$$

- We will show that

$$(\ker \varepsilon_x)\, \Gamma(E) = \ker \varepsilon_x^E \tag{16}$$

- The inclusion $\subseteq$ is obvious

Kernel of ε_x^E

- For $\supseteq$ suppose that $\varepsilon_x^E(s) = 0$. Then taking a trivialization U_x for x we can find sections and write

$$s(x) = a_1(x)s_1(x) + \cdots + a_n(x)s_n(x) \tag{17}$$

on U_x. We can take $a_i \in C(X)$. Then $\underbrace{s - a_1 \cdot s_1 - \cdots - a_n \cdot s_n}_{s'} = 0$ on U_x and by Urysohn's Lemma we can take $\varphi_x \in C(X)$ such that $\varphi_x \cdot s' = 0$ on X. Hence

$$s' = \varphi_x s' + (1 - \varphi_x)s' = (1 - \varphi_x)s' \tag{18}$$

where $(1 - \varphi_x)(x) = 0$. Therefore, $s' \in (\ker \varepsilon_x)\Gamma(E)$ and because $a_i(x) = 0$ since $s(x) = 0$ it follows also that $s_i \in (\ker \varepsilon_x)\Gamma(E)$. We can write $s = s' + \sum a_i s_i$ and therefore $s \in (\ker \varepsilon_x)\Gamma(E)$.

Definition of f

- Because g is a $C(X)$ homomorphism we have that

$$g\left(\ker \varepsilon_x^E\right) = g\left(\left(\ker \varepsilon_x\right)\Gamma(E)\right) \subseteq \left(\ker \varepsilon_x\right)\Gamma(E') = \ker \varepsilon_x^{E'} \tag{19}$$

- Therefore, if $e = s_1(x) = s_2(x)$ for two different sections then $s_1 - s_2 \in \ker \varepsilon_x^E$ and $g(s_1 - s_2) \in \ker \varepsilon_x^{E'}$ which implies that $g(s_1)(x) = g(s_2)(x)$ so we can define

$$f(e) \equiv (gs)(x) \tag{20}$$

- The fiber-wise linearity and $p' \circ f = p$ follow from the properties of g. We only need to check continuity but this is easy because continuity is a local property.

Essential Surjectivity

- Suppose that P is a finitely generated projective module. Then there exists Q such that $P \oplus Q = F$ is free
- Therefore,we can write

$$P \oplus Q \simeq \Gamma\left(X \times \mathbb{C}^k\right) \tag{21}$$

where $X \times \mathbb{C}^k$ is a trivial vector bundle over X. For $s \in \Gamma\left(X \times \mathbb{C}^k\right)$ we can write

$$s = s_P + s_Q \tag{22}$$

and define

$$\begin{aligned} g : \Gamma\left(X \times \mathbb{C}^k\right) &\longrightarrow \Gamma\left(X \times \mathbb{C}^k\right) \\ s &\longrightarrow s_P \end{aligned} \tag{23}$$

Essential Surjectivity

- By our surjectivity there exists $f : X \times \mathbb{C}^k \longrightarrow X \times \mathbb{C}^k$ such that
$$g = \Gamma f \tag{24}$$
- Because Γ is a functor we have
$$\Gamma f^2 = \Gamma f \circ \Gamma f = g^2 = g = \Gamma f \tag{25}$$
and by injectivity it follows that
$$f^2 = f \tag{26}$$
- Taking
$$E = \mathrm{im} f \tag{27}$$
it can be verified that
$$\Gamma E \simeq P \tag{28}$$
.

We have shown the Serre-Swan correspondence!

Hopf Line Bundle

- Consider the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{29}$$

- Define $F : S^2 \subseteq \mathbb{R}^3 \longrightarrow M_2(\mathbb{C})$ by

$$F(x_1, x_2, x_3) = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \tag{30}$$

- The following matrix is idempotent of rank 1

$$e = \frac{1+F}{2} = \frac{1}{2}\begin{pmatrix} 1+x_3 & x_1+ix_2 \\ x_1-ix_2 & 1-x_3 \end{pmatrix} \tag{31}$$

- By the Serre -Swan correspondence it defines complex line bundle called the Hopf line bundle. It is associated to the Hopf Fibration

$$S^1 \longrightarrow S^3 \longrightarrow S^2 \tag{32}$$

Hopf Line Bundle on Quantum Sphere

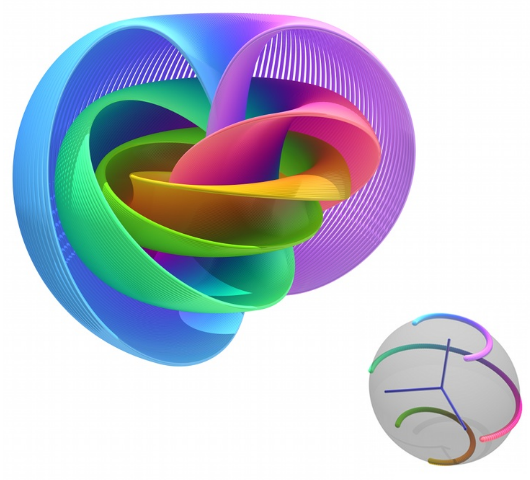
- In the Serre-Swan correspondence, the modules were defined over the commutative ring $C(X)$.
- A non-commutative vector bundle is a finitely generated projective right module P for a non-necessarily commutative algebra $\mathcal{A}$!
- The Podles quantum sphere S_q^2 is the $*$ -algebra over $\mathbb{C}$ generated by a, a^*, b subject to

$$aa^* + q^{-4}b^2 = 1 \quad a^*a + b^2 = 1 \quad ab = q^{-2}ba \quad a^*b = q^2ba^* \tag{33}$$

- The quantum Hopf Bundle is noncommutative line bundle associated to the idempotent

$$e_q = \frac{1}{2}\begin{pmatrix} 1 + q^{-2}b & qa \\ q^{-1}a^* & 1 - b \end{pmatrix} \tag{34}$$

Thank you!



Reference:
https://commons.wikimedia.org/wiki/File:Hopf_Fibration.png