The $SO(3)$ Vortex Equations over Orbifold Riemann Surfaces

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The Elephant in the Room

Image credit: Rappdems

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Some References

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| Bradlow, Garcia-Prada, Non-abelian Monopoles and Vortices>

| Feehan, Leness. Virtual Morse-Bott index, moduli spaces of pairs, and applications to topology of smooth four-manifolds>

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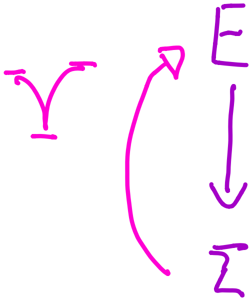
The $SO(3)$ Vortex Equations

$$*F_C^0 - i\left[\Upsilon\Upsilon^* - \frac{1}{2}|\Upsilon|^2 I_E\right] = 0$$

$$\bar{\partial}_C \Upsilon = 0$$



U(2) bundle

C connection on E inducing $C^{\det}$ on $\det E$

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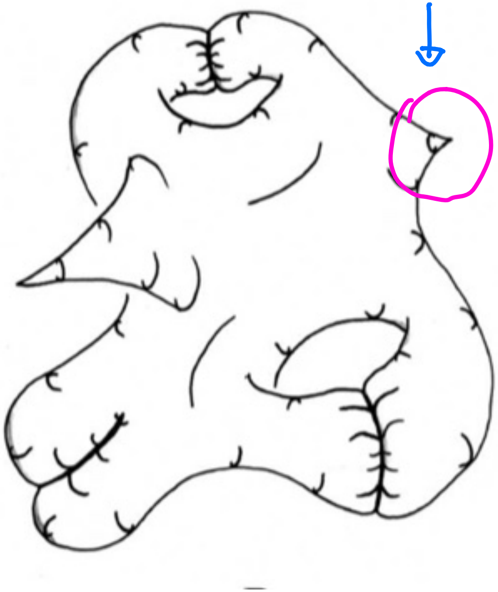
Orbifolds



from Hyde-Ramsden-Robins 14

$(\Sigma, (p_1,a_1), (p_2,a_2), \dots, (p_n,a_n))$

$1 \neq a_i \in \mathbb{Z}^+$

$U_p \simeq \mathbb{C}/(z \to e^{2\pi i/a} z)$

$(z,w) \to (e^{2\pi i/a} z, e^{-2\pi i b/a} w)$

b = isotropy data

$0 \leq b < a$

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Orbifold Bundles

$$L \iff \begin{cases} \text{isotropy data } b_i \\ \underbrace{\deg_B(L)}_{\in \mathbb{Z}} = \underbrace{c_1(L)}_{\in \mathbb{Q}} - \sum \frac{b_i}{a_i} \end{cases}$$

$$E \iff \begin{cases} \det E & \text{iso } b_i \\ \text{isotropy data} & 0 \le b_i^{\pm} < a_i \\ b_i^- + b_i^+ \equiv b_i \mod a_i & \end{cases}$$

$$E \simeq L^- \oplus L^+$$

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Gauge Group and Circle Action

$$e^{i\theta} \cdot [C, \Upsilon] = [C, e^{i\theta}\Upsilon]$$

modulo det 1 gauge group.

Projectively Flat connections:

$$\begin{cases} \Upsilon \equiv 0 \\ F_C^0 = 0 \end{cases}$$

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Abelian Vortices

RIGHT ISOTROPY

$$\begin{cases} E = L \oplus (L^* \otimes \det E) \\ C = C_L \oplus C_{L^* \otimes \det E} \\ \Upsilon = \alpha \oplus 0 \\ \\ \bar\partial_{C_L} \alpha = 0 \\ *F_{C_L} - \frac{i}{2}|\alpha|^2 = *\frac{1}{2} F_{C^{\det}} \end{cases} \implies 0 \le c_1(L) \le \frac{1}{2} c_1(E)$$

If $\alpha \equiv 0$ then $c_1(L) = \frac{c_1(E)}{2}$.

(can't happen if $\det E \cong L_0^{2k+1}$)

a_i coprime $\to \exists L_0$ with $c_1(L_0) = \frac{1}{a_1 \cdots a_n}$

and every L is $L = L_0^k$ for some k

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Moduli Space

Kahler manifold of complex dimension

$$\dim_{\mathbb{C}} \mathcal{M}(\Sigma, E) = g - 1 + c_1(\det E) + n - n_0 - \sum_{i=1}^{n} \frac{b_i^- + b_i^+}{a_i}$$

where $n_0 = \#\{i \mid b_i^- = b_i^+\}$

Smoothness at the flat connections:

$$c_1(E) > 2c_1(K_\Sigma) = 2\left(2g - 2 + n - \sum_{i=1}^{n} \frac{1}{a_i}\right)$$

$\Omega^{1,0}(\Sigma)$

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A Morse-Bott Function

$$\mu(C,\Upsilon) = \frac{1}{2}\|\Upsilon\|^2_{L^2(\Sigma)}$$

critical points of $\mu \Leftrightarrow$ fixed points of S^1 action

$$\mu(C_L,\alpha) = \pi(c_1(E) - 2c_1(L)) \quad \text{at } (C_L,\alpha)$$

$$\text{ind}(E,L) =$$

$$2\,[g-1+c_1(\det E)-2c_1(L)]$$

$$+2\left[\sum_{i|\epsilon_i=1}\frac{b_i^+ - b_i^-}{a_i} + n_- + \sum_{i|\epsilon_i=-1}\frac{b_i^- - b_i^+}{a_i}\right]$$

where

$\epsilon_i = 1$ if $b_i = b_i^+$, $\epsilon_i = -1$ if $b_i = b_i^-$, $n_- = \#\{i \mid b_i = b_i^- < b_i^+\}$

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Spin-u Structures / SO(3) monopoles on 3 manifolds Y

$$V = S \otimes E \simeq (S \otimes L) \otimes (E \otimes L^{-1})$$

$$\downarrow$$

$$Y$$

$$\begin{cases} c_1(\mathfrak{t}) \equiv & c_1(\mathfrak{s}) + c_1(E) \\ w_2(\mathfrak{t}) \equiv & w_2(\mathsf{ad}(E)) \end{cases}$$

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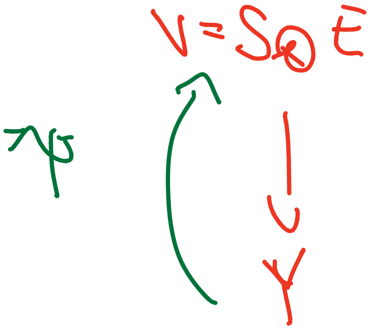
$SO(3)$ Monopoles

$$*F_B^0 + 2\rho^{-1}(\Psi\Psi^*)_{00} = 0$$

$$D_B\Psi = 0$$



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Fixed Points

- Flat connections:

$$F_B^0 = 0$$

- Seiberg-Witten Monopoles:

$\psi = \psi \oplus 0$

$$*F_{B_L} + \rho^{-1}(\psi\psi^*)_0 - \underbrace{\frac{1}{2} * F_{B^{\det}}}_{\omega} = 0$$

$$D_B\psi = 0$$

If $\psi \equiv 0$ then $w_2(\mathrm{ad}E) = 0$.

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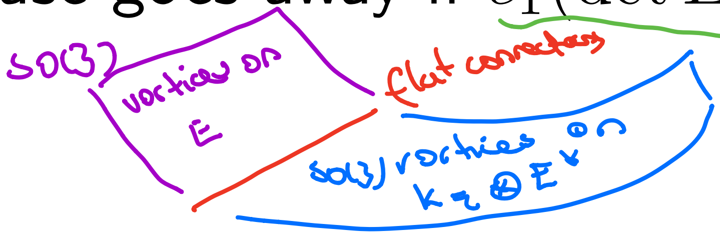
The Case of $Y = S^1 \times \Sigma$

$$V = (\mathbb{C} \oplus K_\Sigma^{-1}) \otimes E = E \oplus (K_\Sigma^{-1} \otimes E)$$

$$\Psi = \alpha \oplus \beta$$

- $\beta \equiv 0$: (B, α) can be identified with an $SO(3)$ vortex on E
- $\alpha \equiv 0$: (B, β) can be identified with an $SO(3)$ vortex on $K_\Sigma \otimes E^*$
- Last case goes away if $c_1(\det E) > 2c_1(K_\Sigma)$



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Food for Thought

http://www.carbslow.com/styled-11/

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$HI^{\#}(Y)$ and $\widetilde{HM}(Y)$

Conjecture [Kronheimer-Mrowka]

$$HI^{\#}(Y) \simeq \widetilde{HM}(Y) \otimes \mathbb{C} \simeq \widehat{HF}(Y) \otimes \mathbb{C}$$

" $= \frac{1}{2} HI(Y \# T^3)$ "

In particular

$HI^{\#}(S^3) = \frac{1}{2} HI(T^3) \simeq$ one dimensional

$T^3 = S^1 \times T^2$

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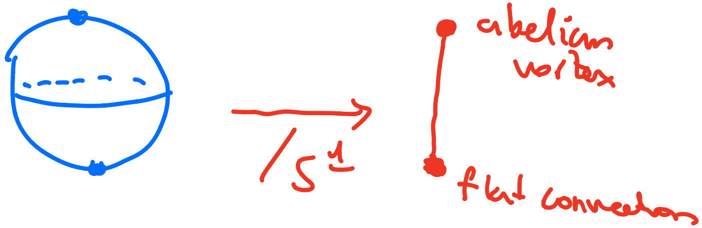
$SO(3)$ Vortices on T^2

$\deg E = 1$

$\dim_{\mathbb{R}} \mathcal{M}(\Sigma, E) = 2$



$HM^{\#}(Y) \equiv HM(Y \# T^3, \omega)$

perturbation term

$HM^{\#}(S^3) = HM(T^3, \omega)$ one dimensional

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Seifert Manifolds

$$\nabla^{\infty} = d \oplus \pi^*(\nabla_{\Sigma}^{LC})$$

$$V_{can} = (\mathbb{C} \oplus \pi^*(K_{\Sigma}^{-1})) \otimes \pi^*(E)$$

L

$\downarrow$

Σ

$Y = S^1(L)$

π

a_i coprime

$Y = S^1(L_0) = \Sigma(a_1, \ldots, a_n)$

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MOY for $SO(3)$ Monopoles

- $\Psi = \alpha \oplus \beta$
- $\beta \equiv 0$: (B, α) can be identified with an $SO(3)$ vortex on E', where $\det E' \simeq \det E$
- $\alpha \equiv 0$: (B, β) can be identified with an $SO(3)$ vortex on $K_\Sigma \otimes E'^*$, where $\det E' \simeq \det E$
- Last case goes away if $c_1(\det E) > 2c_1(K_\Sigma)$

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$\Sigma(2,3,5)$ $\longrightarrow$ $S^2(2,3,5)$

$L_0 = L_0(b_1, b_2, b_3)$

- $a_1 = 2,\ a_2 = 3,\ a_3 = 5$

$$\frac{1}{\prod a_i} - \sum_{i=1}^{n} \frac{b_i}{a_i} = \frac{1}{30} - \frac{15b_1 + 10b_2 + 6b_3}{30} \in \mathbb{Z}$$

$$\Longrightarrow\ b_1 = b_2 = b_3 = 1$$

- $L_0(1,1,1)$ satisfies

$$c_1(L_0) = \frac{1}{30}$$

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$\Sigma(2,3,5)$

$$c_1(E) > 2c_1(K_{S^2(2,3,5)}) = 2\left(0 - 2 + 3 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5}\right) = -\frac{1}{15}$$

So can take:

$$\det E \simeq L_0$$

$$\begin{cases} 0 \leq b_i^- \leq b_i^+ < a_i \\ b_i^- + b_i^+ \equiv 1 \mod a_i \end{cases}$$

$E\,[(b_1^-, b_1^+), (b_2^-, b_2^+), (b_3^-, b_3^+)]$

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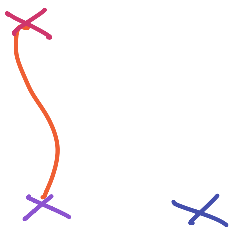
$\Sigma(2,3,5)$, $\det E \simeq L_0(1,1,1)$

Isotropy	$\mathcal{M}^*(\Sigma, E)$	# Flat	# Ab. Vor
$((0,1),(0,1),(0,1))$	$\neq \emptyset$, d. 2	one	1 (ind 2)
$((0,1),(0,1),(2,4))$	$\emptyset$, d. 0	one (iso)	$\emptyset$ (bad iso)
$((0,1),(0,1),(3,3))$	$\emptyset$, d. -2	$\emptyset$	$\emptyset$ (bad iso)
$((0,1),(2,2),(0,1))$	$\emptyset$, d. -2	$\emptyset$	$\emptyset$ (bad iso)
$((0,1),(2,2),(2,4))$	$\emptyset$, d. -4	$\emptyset$	$\emptyset$ (bad iso)
$((0,1),(2,4),(3,3))$	$\emptyset$, d. -4	$\emptyset$	$\emptyset$ (bad iso)



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Casson invariant = " $\frac{1}{2}$ # flat connections "

||

" # SW monopoles "

" # flat connections = 2 # SW monopoles "

$\Sigma(2,3,7)$: $\det E \simeq L_0^5(1,1,2)$

Isotropy	$\mathcal{M}^*(\Sigma, E)$	# Flat	# Ab. Vor
$((0,1),(0,1),(0,2))$	$\neq \emptyset$, d. 2	one	1 (ind 2)
$((0,1),(0,1),(1,1))$	$\emptyset$ d. 0	$\emptyset$	$\emptyset$ (bad iso)
$((0,1),(0,1),(3,6))$	$\emptyset$ d. 0	one	$\emptyset$ (bad iso)
$((0,1),(0,1),(4,5))$	$\emptyset$, d. 0	$\emptyset$	$\emptyset$ (bad iso)
$((0,1),(2,2),(0,2))$	$\emptyset$, d. -2	$\emptyset$	$\emptyset$ (bad iso)
$((0,1),(2,2),(1,1))$	$\emptyset$ d. -4	$\emptyset$	$\emptyset$ (bad iso)
$((0,1),(2,2),(3,6))$	$\emptyset$, d. -4	$\emptyset$	$\emptyset$ (bad iso)
$((0,1),(2,2),(4,5))$	$\emptyset$ d. -4	$\emptyset$	$\emptyset$ (bad iso)

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What is next?

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Thank you!

https://christophergbaker.com/blender-3d/