A Generalization of the Tristram Levine Knot Signatures as a Singular Furuta-Ohta Invariant for Tori

Mariano Echeverria

Completing the (gauge theory) square

$$Y^3, ZHS^3$$

Completing the (gauge theory) square

$$Y^3, ZHS^3$$

$$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$$

Completing the (gauge theory) square

Y^3, ZHS^3 $X, ZH(S^1 \times S^3) + \cdots$

$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$

Completing the (gauge theory) square

Y^3, ZHS^3 $\quad$ $X, ZH(S^1 \times S^3) + \cdots$

$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$ $\quad$ $\lambda_{FO}(X) \sim \#_s\{\pi_1(X) \to SU(2)\}$

Completing the (gauge theory) square

Y^3, ZHS^3

$X, ZH(S^1 \times S^3) + \cdots$

$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$

$\lambda_{FO}(X) \sim \#_s\{\pi_1(X) \to SU(2)\}$

$\lambda_{FO}(S^1 \times Y) = \lambda_C(Y)$

Completing the (gauge theory) square

Y^3, ZHS^3

$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$

$X, ZH(S^1 \times S^3) + \cdots$

$\lambda_{FO}(X) \sim \#_s\{\pi_1(X) \to SU(2)\}$

$\lambda_{FO}(S^1 \times Y) = \lambda_C(Y)$

$K \subset Y^3$

Completing the (gauge theory) square

$$Y^3, ZHS^3 \qquad X, ZH(S^1 \times S^3) + \cdots$$

$$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\} \qquad \lambda_{FO}(X) \sim \#_s\{\pi_1(X) \to SU(2)\}$$

$$\lambda_{FO}(S^1 \times Y) = \lambda_C(Y)$$

$$K \subset Y^3$$

$$\lambda_{CLH}(Y, K, \alpha)$$

Completing the (gauge theory) square

$$Y^3, ZHS^3$$

$$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$$

$$X, ZH(S^1 \times S^3) + \cdots$$

$$\lambda_{FO}(X) \sim \#_s\{\pi_1(X) \to SU(2)\}$$

$$\lambda_{FO}(S^1 \times Y) = \lambda_C(Y)$$

$$K \subset Y^3$$

$$\lambda_{CLH}(Y, K, \alpha)$$

$$\sim \#_s\{\pi_1(Y \backslash K) \to SU(2) \mid$$

$$+ \cdots\}$$

Defining $\lambda_{CLH}(Y, K, \alpha)$

- $H_1(Y \setminus K; \mathbb{Z}) \simeq \mathbb{Z}[\mu_K]$

Defining $\lambda_{CLH}(Y, K, \alpha)$

- $H_1(Y \backslash K; \mathbb{Z}) \simeq \mathbb{Z}[\mu_K]$
- $\rho_{red} : \pi_1(Y \backslash K) \to SU(2) \iff \rho_{red}(\mu_K) \in SU(2)$

Defining $\lambda_{CLH}(Y, K, \alpha)$

- $H_1(Y \backslash K; \mathbb{Z}) \simeq \mathbb{Z}[\mu_K]$
- $\rho_{red} : \pi_1(Y \backslash K) \to SU(2) \iff \rho_{red}(\mu_K) \in SU(2)$
- Up to conjugacy, $\rho_{red,\alpha}(\mu_K) = \begin{pmatrix} e^{-2\pi i \alpha} & 0 \\ 0 & e^{2\pi i \alpha} \end{pmatrix}$

Defining $\lambda_{CLH}(Y, K, \alpha)$

- $H_1(Y \setminus K; \mathbb{Z}) \simeq \mathbb{Z}[\mu_K]$
- $\rho_{red} : \pi_1(Y \setminus K) \to SU(2) \iff \rho_{red}(\mu_K) \in SU(2)$
- Up to conjugacy, $\rho_{red,\alpha}(\mu_K) = \begin{pmatrix} e^{-2\pi i \alpha} & 0 \\ 0 & e^{2\pi i \alpha} \end{pmatrix}$
- For $0 < \alpha < \frac{1}{2}$, consider

$$\mathcal{R}(Y, K, \alpha) = \{\rho : \pi_1(Y \setminus K) \to SU(2) \mid \rho(\mu_K) \sim \rho_{red,\alpha}\}/\sim$$

Defining $\lambda_{CLH}(Y, K, \alpha)$

- $H_1(Y \backslash K; \mathbb{Z}) \simeq \mathbb{Z}[\mu_K]$
- $\rho_{red} : \pi_1(Y \backslash K) \to SU(2) \iff \rho_{red}(\mu_K) \in SU(2)$
- Up to conjugacy, $\rho_{red,\alpha}(\mu_K) = \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}$
- For $0 < \alpha < \frac{1}{2}$, consider

$$\mathcal{R}(Y, K, \alpha) = \{\rho : \pi_1(Y \backslash K) \to SU(2) \mid \rho(\mu_K) \sim \rho_{red,\alpha}\}/\sim$$

- As long as $\triangle_K(e^{-4\pi i\alpha}) \neq 0$, the unique reducible $\rho_{red,\alpha}$ is isolated so $\lambda_{CLH}(Y, K, \alpha)$ can be defined.

Defining $\lambda_{CLH}(Y, K, \alpha)$

- $H_1(Y \backslash K; \mathbb{Z}) \simeq \mathbb{Z}[\mu_K]$
- $\rho_{red} : \pi_1(Y \backslash K) \to SU(2) \iff \rho_{red}(\mu_K) \in SU(2)$
- Up to conjugacy, $\rho_{red,\alpha}(\mu_K) = \begin{pmatrix} e^{-2\pi i \alpha} & 0 \\ 0 & e^{2\pi i \alpha} \end{pmatrix}$
- For $0 < \alpha < \frac{1}{2}$, consider

$$\mathcal{R}(Y, K, \alpha) = \{\rho : \pi_1(Y \backslash K) \to SU(2) \mid \rho(\mu_K) \sim \rho_{red,\alpha}\}/\sim$$

- As long as $\triangle_K(e^{-4\pi i \alpha}) \neq 0$, the unique reducible $\rho_{red,\alpha}$ is isolated so $\lambda_{CLH}(Y, K, \alpha)$ can be defined.
- In this case, Lin (for $\alpha = 1/4$) and Herald (general case) showed that

$$\lambda_{CLH}(Y, K, \alpha) = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i \alpha})$$

Completing the (gauge theory) square

$$Y^3, ZHS^3$$

$$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$$

$$K \subset Y^3$$

$$\lambda_{CLH}(Y, K, \alpha)$$

$$\sim \#_s\{\pi_1(Y \backslash K) \to SU(2) \mid$$

$$\mu_K \to \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}\}$$

$$X, ZH(S^1 \times S^3) + \cdots$$

$$\lambda_{FO}(X) \sim \#_s\{\pi_1(X) \to SU(2)\}$$

$$\lambda_{FO}(S^1 \times Y) = \lambda_C(Y)$$

Completing the (gauge theory) square

$$Y^3, ZHS^3$$

$$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$$

$$X, ZH(S^1 \times S^3) + \cdots$$

$$\lambda_{FO}(X) \sim \#_s\{\pi_1(X) \to SU(2)\}$$

$$\lambda_{FO}(S^1 \times Y) = \lambda_C(Y)$$

$$K \subset Y^3$$

$$\lambda_{CLH}(Y, K, \alpha)$$

$$\sim \#_s\{\pi_1(Y \setminus K) \to SU(2) \mid$$

$$\mu_K \to \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}\}$$

$$T \subset X^4, H_1(T;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z})$$

Completing the (gauge theory) square

$$Y^3, ZHS^3$$

$$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$$

$$X, ZH(S^1 \times S^3) + \cdots$$

$$\lambda_{FO}(X) \sim \#_s\{\pi_1(X) \to SU(2)\}$$

$$\lambda_{FO}(S^1 \times Y) = \lambda_C(Y)$$

$$K \subset Y^3$$

$$\lambda_{CLH}(Y, K, \alpha)$$

$$\sim \#_s\{\pi_1(Y \setminus K) \to SU(2) \mid$$

$$\mu_K \to \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}\}$$

$$T \subset X^4, H_1(T;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z})$$

$$\lambda_{FO}(X, T, \alpha)$$

$$\sim \#_s\{\pi_1(X \setminus T) \to SU(2) \mid$$

$$\mu_T \to \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}$$

Completing the (gauge theory) square

$$Y^3, ZHS^3$$

$$\lambda_C(Y) \sim \#_s\{\pi_1(Y) \to SU(2)\}$$

$$X, ZH(S^1 \times S^3) + \cdots$$

$$\lambda_{FO}(X) \sim \#_s\{\pi_1(X) \to SU(2)\}$$

$$\lambda_{FO}(S^1 \times Y) = \lambda_C(Y)$$

$$K \subset Y^3$$

$$\lambda_{CLH}(Y, K, \alpha)$$

$$\sim \#_s\{\pi_1(Y \setminus K) \to SU(2) \mid \mu_K \to \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}\}$$

$$T \subset X^4, H_1(T;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z})$$

$$\lambda_{FO}(X, T, \alpha)$$

$$\sim \#_s\{\pi_1(X \setminus T) \to SU(2) \mid \mu_T \to \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}$$

$$\lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha)$$

$$= 2\lambda_{CLH}(Y, K, \alpha)$$

Toy example: mapping torus

$$ZHS^3 \qquad\qquad \circlearrowright^{\tau} (Y, K)$$

$$\uparrow_{3-\text{fold cover}}$$

$$ZHS^3 \qquad\qquad (Y', K')$$

Toy example: mapping torus

$$ZHS^3 \qquad \circlearrowright^{\tau} (Y, K) \longrightarrow \qquad (X_\tau, T_\tau) = \frac{[0,1] \times Y}{(0,y) \sim (1, \tau(y))}$$

$$\uparrow_{3-\text{fold cover}}$$

$$ZHS^3 \qquad (Y', K')$$

Toy example: mapping torus

$$ZHS^3 \qquad \circlearrowright^{\tau} (Y, K) \longrightarrow \quad (X_\tau, T_\tau) \quad ZH(S^1 \times S^3)$$

$$\uparrow_{3-\textbf{fold cover}}$$

$$ZHS^3 \qquad (Y', K')$$

$$\lambda_{FO}\left(X_\tau, T_\tau, \frac{1}{5}\right)$$

Toy example: mapping torus

$$
\begin{array}{llcll}
ZHS^3 & \circlearrowright^{\tau} (Y, K) & \longrightarrow & (X_\tau, T_\tau) & ZH(S^1 \times S^3) \\
 & \uparrow_{3-\textbf{fold cover}} & & & \\
ZHS^3 & (Y', K') & & &
\end{array}
$$

$$
\lambda_{FO}\left(X_\tau, T_\tau, \frac{1}{5}\right)
$$

$$
= 2\lambda^{\tau}_{CLH}\left(Y, K, \frac{1}{5}\right)
$$

Toy example: mapping torus

$$ZHS^3 \qquad \circlearrowright^{\tau} (Y, K) \longrightarrow \quad (X_\tau, T_\tau) \quad ZH(S^1 \times S^3)$$

$$\uparrow_{3-\text{fold cover}}$$

$$ZHS^3 \qquad (Y', K')$$

$$\begin{aligned}
&\lambda_{FO}\left(X_\tau, T_\tau, \frac{1}{5}\right)\\
=&2\lambda^{\tau}_{CLH}\left(Y, K, \frac{1}{5}\right)\\
=&2\left(\lambda_{CLH}\left(Y', K', \frac{1}{15}\right) + \lambda_{CLH}\left(Y', K', \frac{6}{15}\right)\right)
\end{aligned}$$

Toy example: mapping torus

$$ZHS^3 \qquad \circlearrowright^{\tau} (Y,K) \longrightarrow \quad (X_\tau, T_\tau) \quad ZH(S^1 \times S^3)$$

$$\uparrow_{3-\textbf{fold cover}}$$

$$ZHS^3 \qquad (Y', K')$$

$$\begin{aligned} &2\lambda^{\tau}_{CLH}\left(Y, K, \frac{1}{5}\right) \\ =&2(\underbrace{\lambda_{CLH}\left(Y', K', \frac{1}{15}\right)}_{3\cdot\frac{1}{15}=\frac{1}{5}} + \underbrace{\lambda_{CLH}\left(Y', K', \frac{6}{15}\right)}) \end{aligned}$$

Toy example: mapping torus

$$ZHS^3 \qquad \circlearrowright^{\tau} (Y,K) \longrightarrow \quad (X_\tau, T_\tau) \quad ZH(S^1 \times S^3)$$

$$\uparrow_{3-\textbf{fold cover}}$$

$$ZHS^3 \qquad (Y', K')$$

$$\begin{aligned} &2\lambda^{\tau}_{CLH}\left(Y, K, \frac{1}{5}\right) \\ =&2(\underbrace{\lambda_{CLH}\left(Y', K', \frac{1}{15}\right)}_{3\cdot\frac{1}{15}=\frac{1}{5}} + \underbrace{\lambda_{CLH}\left(Y', K', \frac{6}{15}\right)}_{3\cdot\frac{6}{15}=\frac{6}{5}=\frac{1}{5}+2\cdot\frac{1}{2}}) \end{aligned}$$

Toy example: mapping torus

$$ZHS^3 \qquad \circlearrowright^{\tau}\ (Y, K) \longrightarrow \quad (X_\tau, T_\tau) \quad ZH(S^1 \times S^3)$$

$$\uparrow_{3-\textbf{fold cover}}$$

$$ZHS^3 \qquad (Y', K')$$

$$\begin{aligned}
&\lambda_{FO}\left(X_\tau, T_\tau, \frac{1}{5}\right) \\
=&2\lambda^{\tau}_{CLH}\left(Y, K, \frac{1}{5}\right) \\
=&2\left(\lambda_{CLH}\left(Y', K', \frac{1}{15}\right) + \lambda_{CLH}\left(Y', K', \frac{6}{15}\right)\right) \\
=&16\lambda_C(Y') + \sigma_{K'}\left(e^{-4\pi i \frac{1}{15}}\right) + \sigma_{K'}\left(e^{-4\pi i \frac{6}{15}}\right)
\end{aligned}$$

Defining additional invariants: $D_0(X, T, k, \alpha)$

Topological classification of $SU(2)$ bundles:

$$\begin{array}{r}(E, E\mid_{\nu(T)} = L \oplus L^{-1}) \\ \downarrow \\ (X, T)\end{array}$$

Defining additional invariants: $D_0(X, T, k, \alpha)$

Topological classification of $SU(2)$ bundles:

$$(E, E\,|_{\nu(T)} = L \oplus L^{-1}) \qquad\qquad k = c_2(E)[X] \in \mathbb{Z}$$

$$\downarrow \iff$$

$$(X, T) \qquad\qquad l = -c_1(L\,|_T)[T] \in \mathbb{Z}$$

Defining additional invariants: $D_0(X, T, k, \alpha)$

$$\dim \mathcal{M}(X, T, k, l, \alpha)$$

Defining additional invariants: $D_0(X, T, k, \alpha)$

$$
\begin{aligned}
&\dim \mathcal{M}(X, T, k, l, \alpha) \\
=&8k - 3(b_2^+ - b^1 + 1) \\
&+ 4l - (2g - 2)
\end{aligned}
$$

Defining additional invariants: $D_0(X, T, k, \alpha)$

$$
\begin{aligned}
&\dim \mathcal{M}(X, T, k, l, \alpha) \\
=&8k - 3(b_2^+ - b^1 + 1) \\
&+ 4l - (2g - 2) \\
=&8k + 4l
\end{aligned}
$$

Defining additional invariants: $D_0(X, T, k, \alpha)$

$$\dim \mathcal{M}(X, T, k, l, \alpha)$$
$$= 8k - 3(b_2^+ - b^1 + 1)$$
$$+ 4l - (2g - 2)$$
$$= 8k + 4l$$

$$\mathcal{E}(X, T, k, l, \alpha)$$
$$= \frac{1}{8\pi^2} \int_{\check{X}} \mathsf{tr}(F_A \wedge F_A)$$

Defining additional invariants: $D_0(X, T, k, \alpha)$

$$\dim \mathcal{M}(X, T, k, l, \alpha) \qquad \mathcal{E}(X, T, k, l, \alpha)$$

$$= 8k - 3(b_2^+ - b^1 + 1) \qquad = \frac{1}{8\pi^2} \int_{\check{X}} \mathsf{tr}(F_A \wedge F_A)$$

$$+ 4l - (2g - 2) \qquad = k + 2\alpha l - \alpha^2 T \cdot T$$

$$= 8k + 4l$$

Defining additional invariants: $D_0(X,T,k,\alpha)$

$$\begin{aligned}
&\dim \mathcal{M}(X,T,k,l,\alpha) \\
=&8k - 3(b_2^+ - b^1 + 1) \\
&+ 4l - (2g-2) \\
=&8k + 4l
\end{aligned}$$

$$\begin{aligned}
&\mathcal{E}(X,T,k,l,\alpha) \\
=&\frac{1}{8\pi^2}\int_{\check{X}} \mathsf{tr}(F_A \wedge F_A) \\
=&k + 2\alpha l - \alpha^2 T \cdot T \\
=&k + 2\alpha l
\end{aligned}$$

Defining additional invariants: $D_0(X, T, k, \alpha)$

$$\begin{aligned}
&\dim \mathcal{M}(X, T, k, l, \alpha) \\
=&8k - 3(b_2^+ - b^1 + 1) \\
&+ 4l - (2g - 2) \\
=&8k + 4l
\end{aligned}$$

$$\begin{aligned}
&\mathcal{E}(X, T, k, l, \alpha) \\
=&\frac{1}{8\pi^2} \int_{\check{X}} \mathrm{tr}(F_A \wedge F_A) \\
=&k + 2\alpha l - \alpha^2 T \cdot T \\
=&k + 2\alpha l \\
=&\frac{\dim \mathcal{M}}{8} + 2\left(\alpha - \frac{1}{4}\right) l
\end{aligned}$$

Defining additional invariants: $D_0(X, T, k, \alpha)$

$$\begin{aligned}
&\dim \mathcal{M}(X, T, k, l, \alpha) \\
=&8k - 3(b_2^+ - b^1 + 1) \\
&+ 4l - (2g - 2) \\
=&8k + 4l
\end{aligned}$$

$$\begin{aligned}
&\mathcal{E}(X, T, k, l, \alpha) \\
=&\frac{1}{8\pi^2} \int_{\check{X}} \mathsf{tr}(F_A \wedge F_A) \\
=&k + 2\alpha l - \alpha^2 T \cdot T \\
=&k + 2\alpha l \\
=&\frac{\dim \mathcal{M}}{8} + 2\left(\alpha - \frac{1}{4}\right) l
\end{aligned}$$

$$\begin{aligned}
&\dim \mathcal{M}(X, T, k, -2k, \alpha) \\
=&0
\end{aligned}$$

Defining additional invariants: $D_0(X,T,k,\alpha)$

$$\begin{aligned}
&\dim \mathcal{M}(X,T,k,l,\alpha) && \mathcal{E}(X,T,k,l,\alpha)\\
=&8k-3(b_2^+-b^1+1) && =\frac{1}{8\pi^2}\int_{\check{X}} \mathsf{tr}(F_A\wedge F_A)\\
&+4l-(2g-2) && =k+2\alpha l-\alpha^2 T\cdot T\\
=&8k+4l && =k+2\alpha l\\
& && =\frac{\dim\mathcal{M}}{8}+2\left(\alpha-\frac{1}{4}\right)l
\end{aligned}$$

$$\begin{aligned}
&\dim \mathcal{M}(X,T,k,-2k,\alpha) && \mathcal{E}(X,T,k,-2k,\alpha)\\
=&0 && =k(1-4\alpha)
\end{aligned}$$

Defining additional invariants: $D_0(X,T,k,\alpha)$

$$\dim \mathcal{M}(X,T,k,l,\alpha)$$
$$=8k-3(b_2^+-b^1+1)$$
$$+4l-(2g-2)$$
$$=8k+4l$$

$$\mathcal{E}(X,T,k,l,\alpha)$$
$$=\frac{1}{8\pi^2}\int_{\check{X}}\mathsf{tr}(F_A\wedge F_A)$$
$$=k+2\alpha l$$
$$=\frac{\dim\mathcal{M}}{8}+2\left(\alpha-\frac{1}{4}\right)l$$

$$\dim \mathcal{M}(X,T,k,-2k,\alpha)$$
$$=0$$

$$\mathcal{E}(X,T,k,-2k,\alpha)$$
$$=k(1-4\alpha)$$

$$D_0(X,T,k,\alpha)=\#_s|\mathcal{M}(X,T,k,-2k,\alpha)|$$

Finishing the square: $HI(Y, K, \alpha)$

Y^3, ZHS^3

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3,\ ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$W : Y \to Y$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

$$W : Y \to Y$$

$$HI(W) : HI(Y) \to HI(Y)$$

Finishing the square: $HI(Y, K, \alpha)$

Y^3, ZHS^3

$\chi(HI(Y)) = 2\lambda_C(Y)$

$2h(Y) =$

$\chi(HI_{red}(Y)) - \chi(HI(Y))$

$W : Y \to Y$

$HI(W) : HI(Y) \to HI(Y)$

$\lambda_{FO}(\bar{W}) =$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3 \qquad W : Y \to Y$$

$$\chi(HI(Y)) = 2\lambda_C(Y) \qquad HI(W) : HI(Y) \to HI(Y)$$

$$2h(Y) = \qquad \lambda_{FO}(\bar{W}) = \frac{1}{2}\mathsf{Lef}(HI(W)) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y)) \qquad \frac{1}{2}\mathsf{Lef}(HI_{red}(W)) - h(\bar{W})$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3 \qquad W : Y \to Y$$

$$\chi(HI(Y)) = 2\lambda_C(Y) \qquad HI(W) : HI(Y) \to HI(Y)$$

$$2h(Y) = \qquad \lambda_{FO}(\bar{W}) = \frac{1}{2}\mathsf{Lef}(HI(W)) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y)) \qquad \frac{1}{2}\mathsf{Lef}(HI_{red}(W)) - h(\bar{W})$$

$$K \subset Y^3$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

$$K \subset Y^3$$

$$\chi_{nov}(HI(Y, K, \alpha))$$

$$W : Y \to Y$$

$$HI(W) : HI(Y) \to HI(Y)$$

$$\lambda_{FO}(\bar{W}) = \frac{1}{2}\mathsf{Lef}(HI(W)) =$$

$$\frac{1}{2}\mathsf{Lef}(HI_{red}(W)) - h(\bar{W})$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

$$K \subset Y^3$$

$$\chi_{nov}(HI(Y, K, \alpha))$$

$$= 2\lambda_{CLH}(Y, K, \alpha) \mid$$

$$W : Y \to Y$$

$$HI(W) : HI(Y) \to HI(Y)$$

$$\lambda_{FO}(\bar{W}) = \frac{1}{2}\textsf{Lef}(HI(W)) =$$

$$\frac{1}{2}\textsf{Lef}(HI_{red}(W)) - h(\bar{W})$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

$$W : Y \to Y$$

$$HI(W) : HI(Y) \to HI(Y)$$

$$\lambda_{FO}(\bar{W}) = \frac{1}{2}\mathbf{Lef}(HI(W)) =$$

$$\frac{1}{2}\mathbf{Lef}(HI_{red}(W)) - h(\bar{W})$$

$$K \subset Y^3$$

$$\chi_{nov}(HI(Y, K, \alpha))$$

$$= 2\lambda_{CLH}(Y, K, \alpha) \mid$$

$$h(Y, K, \alpha) =$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

$$K \subset Y^3$$

$$\chi_{nov}(HI(Y, K, \alpha))$$

$$= 2\lambda_{CLH}(Y, K, \alpha) \mid$$

$$h(Y, K, \alpha) =$$

$$\chi_{nov}(HI^{red}(Y, K, \alpha))$$

$$-\chi_{nov}(HI(Y, K, \alpha))$$

$$W : Y \to Y$$

$$HI(W) : HI(Y) \to HI(Y)$$

$$\lambda_{FO}(\bar{W}) = \frac{1}{2}\mathbf{Lef}(HI(W)) =$$

$$\frac{1}{2}\mathbf{Lef}(HI_{red}(W)) - h(\bar{W})$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

$$K \subset Y^3$$

$$\chi_{nov}(HI(Y, K, \alpha))$$

$$= 2\lambda_{CLH}(Y, K, \alpha) \mid$$

$$h(Y, K, \alpha) =$$

$$\chi_{nov}(HI^{red}(Y, K, \alpha))$$

$$-\chi_{nov}(HI(Y, K, \alpha))$$

$$W : Y \to Y$$

$$HI(W) : HI(Y) \to HI(Y)$$

$$\lambda_{FO}(\bar{W}) = \frac{1}{2}\textbf{Lef}(HI(W)) =$$

$$\frac{1}{2}\textbf{Lef}(HI_{red}(W)) - h(\bar{W})$$

$$(W, \Sigma, \alpha) : (Y, K, \alpha) \to (Y, K, \alpha)$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3, ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

$$K \subset Y^3$$

$$\chi_{nov}(HI(Y, K, \alpha))$$

$$= 2\lambda_{CLH}(Y, K, \alpha) \mid$$

$$h(Y, K, \alpha) =$$

$$\chi_{nov}(HI^{red}(Y, K, \alpha))$$

$$-\chi_{nov}(HI(Y, K, \alpha))$$

$$W : Y \to Y$$

$$HI(W) : HI(Y) \to HI(Y)$$

$$\lambda_{FO}(\bar{W}) = \frac{1}{2}\mathsf{Lef}(HI(W)) =$$

$$\frac{1}{2}\mathsf{Lef}(HI_{red}(W)) - h(\bar{W})$$

$$(W, \Sigma, \alpha) : (Y, K, \alpha) \to (Y, K, \alpha)$$

$$HI(\check{W}, \alpha) : HI(\check{Y}, \alpha) \to HI(\check{Y}, \alpha))$$

Finishing the square: $HI(Y, K, \alpha)$

$$Y^3,\ ZHS^3$$

$$\chi(HI(Y)) = 2\lambda_C(Y)$$

$$2h(Y) =$$

$$\chi(HI_{red}(Y)) - \chi(HI(Y))$$

$$K \subset Y^3$$

$$\chi_{nov}(HI(Y, K, \alpha))$$

$$= 2\lambda_{CLH}(Y, K, \alpha) \mid$$

$$h(Y, K, \alpha) =$$

$$\chi_{nov}(HI^{red}(Y, K, \alpha))$$

$$-\chi_{nov}(HI(Y, K, \alpha))$$

$$W : Y \to Y$$

$$HI(W) : HI(Y) \to HI(Y)$$

$$\lambda_{FO}(\bar{W}) = \frac{1}{2}\mathbf{Lef}(HI(W)) =$$

$$\frac{1}{2}\mathbf{Lef}(HI_{red}(W)) - h(\bar{W})$$

$$(W, \Sigma, \alpha) : (Y, K, \alpha) \to (Y, K, \alpha)$$

$$HI(\check{W}, \alpha) : HI(\check{Y}, \alpha) \to HI(\check{Y}, \alpha))$$

$$\sum_{k \in \mathbb{Z}} D_0(X, T, k, \alpha) T^{-\mathcal{E}(X,T,k,\alpha)} =$$

Finishing the square: $HI(Y, K, \alpha)$

Y^3, ZHS^3	$W : Y \to Y$
$\chi(HI(Y)) = 2\lambda_C(Y)$	$HI(W) : HI(Y) \to HI(Y)$
$2h(Y) =$	$\lambda_{FO}(\bar{W}) = \frac{1}{2}\mathbf{Lef}(HI(W)) =$
$\chi(HI_{red}(Y)) - \chi(HI(Y))$	$\frac{1}{2}\mathbf{Lef}(HI_{red}(W)) - h(\bar{W})$
$K \subset Y^3$	$(W, \Sigma, \alpha) : (Y, K, \alpha) \to (Y, K, \alpha)$
$\chi_{nov}(HI(Y, K, \alpha))$	$HI(\check{W}, \alpha) : HI(\check{Y}, \alpha) \to HI(\check{Y}, \alpha))$
$= 2\lambda_{CLH}(Y, K, \alpha) \mid$	
$h(Y, K, \alpha) =$	$\sum_{k \in \mathbb{Z}} D_0(X, T, k, \alpha) T^{-\mathcal{E}(X,T,k,\alpha)} =$
$\chi_{nov}(HI^{red}(Y, K, \alpha))$	$2\mathbf{Lef}(HI(W, \Sigma, \alpha)) =$
$-\chi_{nov}(HI(Y, K, \alpha))$	

Finishing the square: $HI(Y, K, \alpha)$

$Y^3,\ ZHS^3$	$W : Y \to Y$
$\chi(HI(Y)) = 2\lambda_C(Y)$	$HI(W) : HI(Y) \to HI(Y)$
$2h(Y) =$	$\lambda_{FO}(\bar{W}) = \frac{1}{2}\mathbf{Lef}(HI(W)) =$
$\chi(HI_{red}(Y)) - \chi(HI(Y))$	$\frac{1}{2}\mathbf{Lef}(HI_{red}(W)) - h(\bar{W})$
$K \subset Y^3$	$(W, \Sigma, \alpha) : (Y, K, \alpha) \to (Y, K, \alpha)$
$\chi_{nov}(HI(Y, K, \alpha))$	$HI(\check{W}, \alpha) : HI(\check{Y}, \alpha) \to HI(\check{Y}, \alpha))$
$= 2\lambda_{CLH}(Y, K, \alpha)\ \vert$	
$h(Y, K, \alpha) =$	$\sum_{k \in \mathbb{Z}} D_0(X, T, k, \alpha) T^{-\mathcal{E}(X,T,k,\alpha)} =$
$\chi_{nov}(HI^{red}(Y, K, \alpha))$	$2\mathbf{Lef}(HI(W, \Sigma, \alpha)) =$
$-\chi_{nov}(HI(Y, K, \alpha))$	$2\mathbf{Lef}(HI_{red}(W, \Sigma, \alpha)) - 2h(Y, K, \alpha)$

Thank you!