A (Somewhat Brief) Introduction to Dirac Operators

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Dirac Operators and Mathematics

$$
\begin{tikzcd}
 & \text{Mathematical Physics} \arrow[dl] & \\
\text{Clifford Algebras} \arrow[dr] & \text{Dirac Operator} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \text{Elliptic Operators} \arrow[ul] \\
 & \text{Differential Geometry} \arrow[ur] &
\end{tikzcd}
$$

The Dirac Equation

- In special relativity we have the famous relationship

$$E = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2c^4} \quad (1)$$

- In Quantum Mechanics one "quantizes" the previous equation by turning, E, p_x, p_y, p_z into differential operators

$$E \longrightarrow i\frac{\partial}{\partial t} \quad p_x \longrightarrow -i\frac{\partial}{\partial x} \quad p_y \longrightarrow -i\frac{\partial}{\partial y} \quad p_z \longrightarrow -i\frac{\partial}{\partial z}$$

- Equation 1 becomes

$$i\frac{\partial}{\partial t} = \sqrt{-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + m^2c^4}$$

Dirac Operator in 1d

Goal: find an operator $\mathcal{D}$ such that

$$\mathcal{D}^2 = -\frac{d^2}{dx^2}$$

Dirac Operator in 1d

If you know complex numbers the previous problem is not too difficult, just take

$$\mathcal{D} = i\frac{d}{dx}$$

Formal Self-Adjointness

Assume, that $f, g \in C^\infty(\mathbb{R}, \mathbb{C})$ have compact support; then integration by parts says that

$$
\begin{aligned}
< \mathcal{D}f, g >_{L^2} &= \int_{\mathbb{R}} \left(i \frac{df(x)}{dx} \right) \bar{g}(x) dx \\
&= \left(if(x)\bar{g}(x) \right)\Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} if(x) \frac{d\bar{g}(x)}{dx} dx \\
&= \int_{\mathbb{R}} f(x) \overline{\left(i \frac{dg}{dx} \right)} dx \\
&= < f, \mathcal{D}g >_{L^2}
\end{aligned}
$$

so the Dirac operator in 1d is formally self-adjoint!

Dirac Operator in 2d

Goal: find an operator $\mathcal{D}$ such that

$$\mathcal{D}^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

Dirac Operator in 2d

Inspired in the 1d case, we use the following "ansatz"

$$\mathcal{D} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$$

where a, b are some constants. Since

$$\mathcal{D}^2 = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = a^2\frac{\partial^2}{\partial x^2} + (ab + ba)\frac{\partial^2}{\partial x \partial y} + b^2\frac{\partial^2}{\partial y^2}$$

so we need

$$\begin{cases} a^2 = -1 \\ ab + ba = 0 \\ b^2 = -1 \end{cases}$$

In particular, a and b must anticommute!

Dirac Operator in 2d

It is not hard to check that

$$a = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

satisfy $a^2 = b^2 = -Id$ and $ab = -ba$. Therefore, our Dirac Operator $\mathcal{D} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ in 2d is

$$\mathcal{D} = \begin{pmatrix} 0 & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & 0 \end{pmatrix}$$

and so

$$\mathcal{D}^2 = \begin{pmatrix} -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \end{pmatrix} = \left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) Id$$

The Cauchy Riemann Operator

We can rewrite the Dirac operator as

$$\mathcal{D} \;=\; 2i\left(\begin{array}{cc} 0 & \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) & 0 \end{array}\right) \;=\; 2i\left(\begin{array}{cc} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{array}\right)$$

Therefore

$$\left(\begin{array}{c} \psi \\ \phi \end{array}\right) \in \ker \mathcal{D} \longleftrightarrow \psi \text{ is holomorphic and } \phi \text{ is antiholomorphic}$$

Formal Self-Adjointness

If we write

$$\mathcal{D} = \begin{pmatrix} 0 & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{D} \\ D & 0 \end{pmatrix}$$

using Green's Theorem it can be checked that for $f, g \in C^\infty(\mathbb{R}^2, \mathbb{C})$

$$< Df, g >_{L^2} = < f, \tilde{D}g >_{L^2}$$

so that $\tilde{D}$ is the formal adjoint of D. So we can write

$$\mathcal{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \tag{2}$$

which proves again that the Dirac operator is formally self-adjoint!

Spoiler Alert: in even dimensions we will always have a decomposition for the Dirac operator like previous one!

Dirac Operator in 3d

Just as we did for one and two dimensions, if we take

$$\mathcal{D} = c_1 \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y} + c_3 \frac{\partial}{\partial z}$$

then $\mathcal{D}^2 = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ if and only if

$$\begin{cases} c_1^2 = c_2^2 = c_3^2 = -1 \\ c_1 c_2 + c_2 c_1 = c_1 c_3 + c_3 c_1 = c_2 c_3 + c_3 c_2 = 0 \end{cases}$$

The previous system of equations is satisfied by Hamilton's Quaternions, i.e, we can take

$$\mathcal{D} = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

Dirac Operator on $\mathbb{R}^n$ and Clifford Algebras

For $\mathbb{R}^n$, the Dirac Operator (for the standard inner product) will be

$$\mathcal{D} = \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2} + \cdots + \gamma_n \frac{\partial}{\partial x_n}$$

where the γ_i are the generators for the Clifford Algebra of $\mathbb{R}^n$, i.e,

$$\begin{cases} \gamma_i^2 = -1 \\ \{\gamma_i, \gamma_j\} = 0 \quad \text{if } i \neq j \end{cases}$$

What About the Dirac Equation?

We were trying to find $\mathcal{D}$ such that

$$\mathcal{D}^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

In fact we can take

$$\mathcal{D} = \begin{pmatrix} \partial_t & 0 & \partial_z & \partial_x - i\partial_y \\ 0 & \partial_t & \partial_x + i\partial_y & -\partial_z \\ -\partial_z & -\partial_x + i\partial_y & -\partial_t & 0 \\ -\partial_x - i\partial_y & \partial_z & 0 & -\partial_t \end{pmatrix}$$

The Dirac equation was used to predict the existence of antiparticles!

What About Other Spaces?

- What do we mean by the Laplacian on an arbitrary manifold M? Does it always exist?
- What do we mean by the Dirac Operator on an arbitrary manifold M? Does it always exist?

Ingredient 1: Fourier Transform

Recall that for a sufficiently well behaved function $u : \mathbb{R} \longrightarrow \mathbb{R}$ we can define its Fourier transform $\mathcal{F}u : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\mathcal{F}u(x) = \hat{u}(p) = \int_{\mathbb{R}} e^{-2\pi i p x} u(x) dx$$

Modulo some constants, differentiation becomes multiplication in that

$$\frac{d\hat{u}}{dp} = \int_{\mathbb{R}} (-2\pi i x)\, e^{-2\pi i p x} \hat{u}(p) dp = [\widehat{-2\pi i x u(x)}] = \mathcal{F}\left(-2\pi i x u(x)\right)$$

$$\begin{tikzcd}
u \arrow[r, "-2\pi i x \cdot"] \arrow[d, "\mathcal{F}"'] & -2\pi i u \arrow[d, "\mathcal{F}"] \\
\hat{u} \arrow[r, "\frac{d}{dp}"] & \frac{d\hat{u}}{dp}
\end{tikzcd}$$

Ingredient 2: Local Coordinates

On a general manifold M we don't have global coordinates (think of a sphere), however, we can use local coordinates $x_1, \cdots, x_n$ to describe it. In particular, a linear differential operator L of order m can be described locally as an operator

$$L : C^\infty(M; \mathbb{R}^q) \longrightarrow C^\infty(M; \mathbb{R}^p)$$

given by

$$Lu = \sum_{|\alpha| \le m} a_\alpha(x) \underbrace{\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}}_{D^\alpha} u$$

where $u = (u_1, \cdots, u_q) : M \longrightarrow \mathbb{R}^q$ and the $a_\alpha(x)$ are $p \times q$ matrices

Symbol of a Differential Operator

For our differential operator

$$Lu = \sum_{|\alpha|\leq m} a_\alpha(x) \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u$$

we define its (leading) symbol by replacing the partial derivatives by "momenta"

$$\sigma_L(x,p) \equiv \sum_{|\alpha|=m} a_\alpha(x) p_1^{\alpha_1} \cdots p_n^{\alpha_n}$$

Example on $M = \mathbb{R}^3$

$$L : C^\infty(M, \mathbb{R}^3) \longrightarrow C^\infty(M, \mathbb{R}^2)$$
$$(u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x})) \longrightarrow \left(x_1 x_2 \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial u_3}{\partial x_1} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3}, \frac{\partial^2 u_3}{\partial x_1 \partial x_3} \right)$$

For the (leading) symbol keep the highest order terms so we work with

$$\left(x_1 x_2 \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3}, \frac{\partial^2 u_3}{\partial x_1 \partial x_3} \right) = \left(x_1 x_2 \frac{\partial^2 u_1}{\partial x_1^2}, 0 \right) + \left(\frac{\partial^2 u_2}{\partial x_1 \partial x_3}, \frac{\partial^2 u_3}{\partial x_1 \partial x_3} \right)$$

$$L(u_1, u_2, u_3) = \left[\begin{pmatrix} x_1 x_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial^2}{\partial x_1^2} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_3} \right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\sigma_L(x_1, x_2, x_3, p_1, p_2, p_3) = \begin{pmatrix} x_1 x_2 p_1^2 & p_1 p_3 & 0 \\ 0 & 0 & p_1 p_3 \end{pmatrix}$$

Gradient on $M = \mathbb{R}^3$

$$\nabla : C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R}^3)$$
$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$u(x_1,x_2,x_3) \longrightarrow \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix}0\\1\\0\end{pmatrix}\frac{\partial}{\partial y} + \begin{pmatrix}0\\0\\1\end{pmatrix}\frac{\partial}{\partial z}\right] u$$

Its symbol is

$$\sigma_\nabla(\mathbf{x},\mathbf{p}) = \begin{pmatrix}1\\0\\0\end{pmatrix} p_1 + \begin{pmatrix}0\\1\\0\end{pmatrix} p_2 + \begin{pmatrix}0\\0\\1\end{pmatrix} p_3 = \begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix}$$

Curl on $M = \mathbb{R}^3$

$$\mathrm{curl} : C^\infty(M, \mathbb{R}^3) \longrightarrow C^\infty(M, \mathbb{R}^3)$$
$$\nabla \times (u_1, u_2, u_3) = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right)$$

$$(u_1, u_2, u_3) \to$$
$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial y} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial z} \right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Its symbol is

$$\sigma_{\mathrm{curl}}(\mathbf{x}, \mathbf{p}) = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix}$$

Divergence on $M = \mathbb{R}^3$

$$\begin{gathered}\mathrm{div} : C^\infty(M, \mathbb{R}^3) \longrightarrow C^\infty(M, \mathbb{R}) \\ \nabla \cdot (u_1, u_2, u_3) = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\end{gathered}$$

$$(u_1, u_2, u_3) \to$$

$$\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{\partial}{\partial y} + \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial z}\right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Its symbol is

$$\sigma_{\mathrm{div}}(\mathbf{x}, \mathbf{p}) = \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$$

Generalized Laplacian and Dirac Type Operator

Suppose that $\mathcal{E}$ is a vector bundle over a Riemannian manifold M.

- A Generalized Laplacian on $\mathcal{E}$ is a second order differential operator $\triangle$ such that

$$\sigma_{\triangle}(x, \mathbf{p}) = -\|\mathbf{p}\|^2$$

- An Operator of Dirac Type on $\mathcal{E}$ is a first order differential operator $\mathcal{D}$ such that

$$\sigma_{\mathcal{D}^*\mathcal{D}}(x, \mathbf{p}) = -\|\mathbf{p}\|^2$$

  i.e, the symbol of its "square" acts in the same way as the symbol of a generalized Laplacian

An Old Friend

Recall that in $\mathbb{R}^2$ our Dirac operator was

$$\mathcal{D} = \begin{pmatrix} 0 & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

Since

$$\sigma_D(\mathbf{x}, \mathbf{p}) = ip_1 - p_2 \qquad \sigma_{D^*}(\mathbf{x}, \mathbf{p}) = ip_1 + p_2$$

then

$$\sigma_{D^*D}(\mathbf{x}, \mathbf{p}) = \sigma_{D^*}(\mathbf{x}, \mathbf{p}) \circ \sigma_D(\mathbf{x}, \mathbf{p}) = (ip_1 + p_2)(ip_1 - p_2) = -p_1^2 - p_2^2 = -\|\mathbf{p}\|^2$$

so D is an operator of Dirac type.

Vector Operations on $M = \mathbb{R}^3$ and de Rham Cohomology

$$0 \to C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla} C^\infty_{\mathbb{R}^3}(\mathbb{R}^3) \xrightarrow{\nabla\times} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla\cdot} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \to 0$$

From vector calculus we know that

$$\begin{cases} \nabla \times (\nabla\phi) = \mathbf{0} & \text{curl(grad)=0} \\ \\ \nabla \cdot (\nabla \times \mathbf{E}) = 0 & \text{div(curl)=0} \end{cases}$$

therefore we can define a cohomology $H^*_{DR}(\mathbb{R}^3)$!

Computing $H^0_{DR}(\mathbb{R}^3)$

$$0 \to C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla} C^\infty_{\mathbb{R}^3}(\mathbb{R}^3) \xrightarrow{\nabla\times} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla\cdot} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \to 0$$

$$H^0_{DR}(\mathbb{R}^3) = \ker \nabla = \{\phi : \mathbb{R}^3 \longrightarrow \mathbb{R} \mid \nabla\phi \equiv 0\}$$

Now, $\nabla\phi \equiv 0$ if and only if ϕ is a constant scalar field. Since there is a constant scalar field for each real number we have that

$$H^0_{DR}(\mathbb{R}^3) \simeq \mathbb{R}$$

Computing $H^1_{DR}(\mathbb{R}^3)$

$$0 \to C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla} C^\infty_{\mathbb{R}^3}(\mathbb{R}^3) \xrightarrow{\nabla\times} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla\cdot} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \to 0$$

$$H^1_{DR}(\mathbb{R}^3) = \frac{\ker(\text{curl})}{\text{im}\,(\text{grad})} = \{[\mathbf{E}] \mid \nabla \times \mathbf{E} = \mathbf{0} \text{ and } \mathbf{E} \sim \mathbf{E}' \text{ iff } \mathbf{E}' = \mathbf{E} + \nabla\phi\}$$

From Advanced Calculus we know that every irrotational vector field $\mathbf{E}$ is conservative, i.e, $\mathbf{E} = \nabla\phi$ so $\mathbf{E} \in [\mathbf{0}]$ and hence

$$H^1_{DR}(\mathbb{R}^3) = 0$$

Computing $H^2_{DR}(\mathbb{R}^3)$

$$0 \to C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla} C^\infty_{\mathbb{R}^3}(\mathbb{R}^3) \xrightarrow{\nabla\times} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla\cdot} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \to 0$$

$$H^2_{DR}(\mathbb{R}^3) = \frac{\ker(\mathrm{div})}{\mathrm{im}(\mathrm{curl})} = \{[\mathbf{B}] \mid \nabla \cdot \mathbf{B} = 0 \text{ and } \mathbf{B} \sim \mathbf{B}' \text{ iff } \mathbf{B}' = \mathbf{B} + \nabla \times \mathbf{A}\}$$

From Electromagnetism/Vector Calculus we know that every solenoidal vector field $\mathbf{B}$ has a vector potential $\mathbf{A}$, i.e, $\mathbf{B} = \nabla \times \mathbf{A}$ so $\mathbf{B} \in [\mathbf{0}]$ and hence

$$H^2_{DR}(\mathbb{R}^3) = 0$$

Computing $H^3_{DR}(\mathbb{R}^3)$

$$0 \to C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla} C^\infty_{\mathbb{R}^3}(\mathbb{R}^3) \xrightarrow{\nabla\times} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \xrightarrow{\nabla\cdot} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \to 0$$

$H^3_{DR}(\mathbb{R}^3) = \text{coker}(\text{div}) = \{[f] \mid f : \mathbb{R}^3 \longrightarrow \mathbb{R} \text{ and } f \sim g \text{ iff } f = g + \nabla \cdot \mathbf{B}$

Since $f(x,y,z) = \nabla \cdot \left(\int_0^x f(t,y,z)dt, 0, 0\right)$ we have $f \in [0]$ so

$$H^3_{DR}(\mathbb{R}^3) = 0$$

Differential Equations and Topology: de Rham's Theorem

De Rham's Theorem: the de Rham cohomology is isomorphic to the cohomology of the manifold, i.e,

$$H^*_{DR}(M) \simeq H^*(M)$$

In particular we can compute the Euler characteristic of the manifold as

$$\chi(M) = \sum_{i=0}^{\dim M} (-1)^i \dim H^i_{DR}(M)$$

which means that the topology of the space restricts the dimensions of the space of solutions of certain differential equations (Laws of Physics) on our manifold!

The Hodge Laplacian

$$0 \rightleftarrows C^\infty_{\mathbb{R}^3}(\mathbb{R}) \underset{-\nabla\cdot}{\overset{\nabla}{\rightleftarrows}} C^\infty_{\mathbb{R}^3}(\mathbb{R}^3) \underset{\nabla\times}{\overset{\nabla\times}{\rightleftarrows}} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \underset{-\nabla}{\overset{\nabla\cdot}{\rightleftarrows}} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \rightleftarrows 0$$

We can define the Hodge Laplacians

$$\triangle_0 : C^\infty_{\mathbb{R}^3}(\mathbb{R}) \longrightarrow C^\infty_{\mathbb{R}^3}(\mathbb{R}) \qquad \triangle_1 : C^\infty_{\mathbb{R}^3}(\mathbb{R}^3) \longrightarrow C^\infty_{\mathbb{R}^3}(\mathbb{R}^3)$$

$$\triangle_0(f) = -\nabla \cdot (\nabla f) \qquad \triangle_1(\mathbf{u}) = \nabla(-\nabla \cdot \mathbf{u}) + \nabla \times (\nabla \times \mathbf{u})$$

$$= -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) \qquad = (\triangle_0(u_1), \triangle_0(u_2), \triangle_0(u_3))$$

The Hodge Operator

We can combine them to define a Laplacian on

$$\Omega^*_M = \left[C^\infty_{\mathbb{R}^3}(\mathbb{R}) \oplus C^\infty_{\mathbb{R}^3}(\mathbb{R}^3)\right] \oplus \left[C^\infty_{\mathbb{R}^3}(\mathbb{R}) \oplus C^\infty_{\mathbb{R}^3}(\mathbb{R}^3)\right]$$

Define the Hodge- Laplacian

$$\triangle_H : \Omega^*_M \longrightarrow \Omega^*_M$$
$$\triangle_H \begin{pmatrix} (f, \mathbf{u}) \\ (g, \mathbf{v}) \end{pmatrix} = \begin{pmatrix} (\triangle_0 f, \triangle_1 \mathbf{u}) \\ (\triangle_0 g, \triangle_1 \mathbf{v}) \end{pmatrix}$$

Does it have a Dirac operator?

Observe that

$$\mathrm{rank}\,\Omega^*_M = 8 = 2^3 = 2^{\dim M}$$

The Hodge-Dirac Operator

$$0 \rightleftarrows C^\infty_{\mathbb{R}^3}(\mathbb{R}) \underset{-\nabla\cdot}{\overset{\nabla}{\rightleftarrows}} C^\infty_{\mathbb{R}^3}(\mathbb{R}^3) \underset{\nabla\times}{\overset{\nabla\times}{\rightleftarrows}} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \underset{-\nabla}{\overset{\nabla\cdot}{\rightleftarrows}} C^\infty_{\mathbb{R}^3}(\mathbb{R}) \rightleftarrows 0$$

f	$\mathbf{v}$	$\mathbf{u}$	g
0	1	2	3

$$\mathcal{D} : \Omega^*_M \longrightarrow \Omega^*_M$$

$$\mathcal{D}\begin{pmatrix} (f,\mathbf{u}) \\ (g,\mathbf{v}) \end{pmatrix} = \begin{pmatrix} (-\nabla\cdot\mathbf{v},\ -\nabla g + \nabla\times\mathbf{v}) \\ (\nabla\cdot\mathbf{u},\ \nabla f + \nabla\times\mathbf{u}) \end{pmatrix} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}\begin{pmatrix} (f,\mathbf{u}) \\ (g,\mathbf{v}) \end{pmatrix}$$

$$\begin{cases} D(f,\mathbf{u}) = (\nabla\cdot\mathbf{u},\ \nabla f + \nabla\times\mathbf{u}) \\ D^*(g,\mathbf{v}) = (-\nabla\cdot\mathbf{v},\ -\nabla g + \nabla\times\mathbf{v}) \end{cases}$$

$$\mathcal{D}^2\begin{pmatrix} (f,\mathbf{u}) \\ (g,\mathbf{v}) \end{pmatrix} = \triangle_H\begin{pmatrix} (f,\mathbf{u}) \\ (g,\mathbf{v}) \end{pmatrix}$$

Therefore we found a square root for the Laplacian!

"Index" Hodge-Dirac Operator on $M = \mathbb{R}^3$

We have that

$$\ker D = \{(f, \mathbf{u}) \mid \nabla \cdot \mathbf{u} = 0 \text{ and } \nabla f = -\nabla \times \mathbf{u}\}$$

$$\ker D^* = \{(g, \mathbf{v}) \mid \nabla \cdot \mathbf{v} = 0 \text{ and } \nabla g = \nabla \times \mathbf{v}\}$$

and $(f, \mathbf{u}) \longrightarrow (-f, \mathbf{u})$ gives a bijection between $\ker D$ and $\ker D^*$.

$$\text{index} D \quad \equiv \quad \dim\ker D - \dim\ker D^* \stackrel{?}{=} 0$$

The previous calculation fails because $\ker D$ and $\ker D^*$ are infinite dimensional! However, if we run the same argument on a compact, oriented, Riemannian manifold it can be show that

$$\text{index} D = \chi(M)$$

so an analytical quantity (the index) is determined by a topological quantity (Euler Characteristic)!

Fixing the problem: Fredholm Operators

A bounded linear operator $T : E \longrightarrow F$ between Banach spaces is called Fredholm if it has finite dimensional kernel and cokernel. We can define its index by

$$\operatorname{index} T \equiv \dim \ker T - \dim \operatorname{coker} T$$

$\sim\sim\sim$: If E, F are finite dimensional vector spaces then by the rank-nullity theorem ($\dim E = \dim \ker T + \dim \operatorname{im} T$) the index is independent of the operator since

$$\operatorname{index} T = \dim E - \dim F$$

$\sim\sim\sim$: In infinite dimensions the index can depend on the operator. For example, in l^2 we have

$$\text{shift}^+(c_0, c_1, c_2, \cdots) = (0, c_0, c_1, c_2, \cdots,) \qquad \operatorname{ind}\left(\text{shift}^+\right) = -1$$

$$\text{shift}^-(c_0, c_1, c_2, \cdots) = (c_1, c_2, \cdots,) \qquad \operatorname{ind}(\text{shift}^-) = +1$$

Fixing the problem: Elliptic Operators

An operator D is elliptic if

$$\mathbf{p} \neq \mathbf{0} \longrightarrow \sigma_D(x, \mathbf{p}) \text{ is invertible}$$

For example, the symbol for $D(f, \mathbf{u}) = (\nabla \cdot \mathbf{u},\ \nabla f + \nabla \times \mathbf{u})$ is

$$\sigma_D(\mathbf{x}, \mathbf{p}) = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & 0 & -p_3 & p_2 \\ p_2 & p_3 & 0 & -p_1 \\ p_3 & -p_2 & p_1 & 0 \end{pmatrix}$$

and since $\det \sigma_D(\mathbf{x}, \mathbf{p}) = -(p_1^2 + p_2^2 + p_3^2)^2$ we see that whenever $(p_1, p_2, p_3) \neq (0, 0, 0)$ the matrix $\sigma_D(\mathbf{x}, \mathbf{p})$ is invertible, i.e, D is an elliptic operator!

The "wave operator"

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

has symbol

$$\sigma_{\Box}(\mathbf{x}, \mathbf{p}) = p_0^2 - p_1^2 - p_2^2 - p_3^2$$

and it clearly vanishes on the "light cones"
$p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0$

The "heat kernel operator"

$$\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

has symbol

$$\sigma_H(\mathbf{x}, \mathbf{p}) = -p_1^2 - p_2^2 - p_3^2$$

and it clearly vanishes whenever $p_1 = p_2 = p_3 = 0$ and $p_0 \in \mathbb{R}$.

Why do we care about Elliptic Operators on a compact manifold?

1) The operators of Dirac type are always elliptic
2) Over a compact manifold M, being elliptic implies being Fredholm
3) Fredholm operators are very stable under perturbations, which suggests that the index of a Fredholm operator might be computed via topological quantities

The Atiyah-Singer Theorem

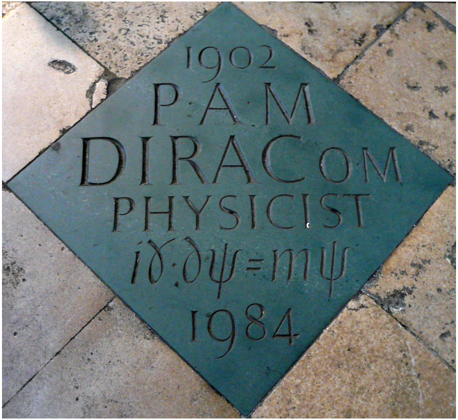
$$\text{index}\mathcal{D} \overset{\text{AS}}{=} \text{topological stuff!}$$

where by “topological stuff” we mean certain characteristic classes associated to the manifold and the vector bundle on which the Dirac operator acts.

What I Left Out

$$\begin{tikzcd}
 & \text{Harmonic Analysis} & \\
\text{Lie Groups and Algebras} & \text{Dirac Operators} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \text{Noncommutative Geometry} \\
 & \text{K-Theory} &
\end{tikzcd}$$

Thank you!



Source:
https://commons.wikimedia.org/wiki/File:Dirac%27s_commemorative_marker.jpg