# Clifford, Spin and All That

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# Clifford Algebras and Mathematics

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\begin{array}{ccccc}
 & & \begin{array}{c}\text{harmonic analysis}\\ \text{mathematical physics}\end{array} & & \\
 & & \uparrow & & \\
\begin{array}{c}\text{Lie Groups}\\ \text{and Algebras}\end{array} & \longleftarrow & \text{Clifford Algebras} & \longrightarrow & \begin{array}{c}\text{differential}\\ \text{geometry}\end{array} \\
 & & \downarrow & & \\
 & & \text{K-theory} & &
\end{array}
\tag{1}
$$

# The Dirac Equation

- In special relativity we have the (famous) relationship

$$E = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2} \tag{2}$$

- In Quantum Mechanics one "quantizes" the previous equation by turning, $E, p_x, p_y, p_z$ into differential operators

$$E \longrightarrow i\frac{\partial}{\partial t} \quad p_x \longrightarrow -i\frac{\partial}{\partial x} \quad p_y \longrightarrow -i\frac{\partial}{\partial y} \quad p_z \longrightarrow -i\frac{\partial}{\partial z} \tag{3}$$

- Equation 2 becomes

$$i\frac{\partial}{\partial t} = \sqrt{-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + m^2} \tag{4}$$

# Square Root for the Laplacian

- One might try first to find first a square root for the Laplacian $\triangle = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$
- $\sqrt{\triangle}$ should be a linear differential operator of first order

$$D = \sum_{i=1}^{3} c_i \frac{\partial}{\partial x_i} \tag{5}$$

and from the condition $D^2 = -\triangle$ we find that the $c_i$ must satisfy (formal manipulation)

$$\begin{cases} c_1^2 = c_2^2 = c_3^2 = -1 \\ c_1 c_2 + c_2 c_1 = c_1 c_3 + c_3 c_1 = c_2 c_3 + c_3 c_2 = 0 \end{cases} \tag{6}$$

- $D$ is called the Dirac Operator

# Pauli Matrices and Clifford Algebras

- Clearly the previous relations can't be satisfied by ordinary numbers (real or complex)
- A representation for the previous relations are the matrices

$$c_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad c_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad c_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{7}$$

- Since we may associate to $-\triangle = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ the quadratic form $q(\mathbf{x}) = -x_1^2 - x_2^2 - x_3^2$ on $\mathbb{R}^3$ we may restate our problem as:

Given a quadratic form $q(x)$ defined on a vector space of dimension $d$ is it possible to embed the vector space $V$ into an algebra $A$ in such a way that for elements $e_1, \cdots, e_d \in V$ we have $e_i^2 = q(e_i) \cdot 1_A$?

# Hamilton's Quaternions

- After many attempts Hamilton found the quaternions $a + ib + jc + kd$ subject to

$$\begin{cases} i^2 = j^2 = k^2 = -1 \\ ij + ji = ik + ki = jk + kj = 0 \end{cases} \tag{8}$$

- To close the algebra, he introduced also the relations

$$ij = k \qquad jk = i \qquad ki = j \tag{9}$$

- The Clifford Algebra can be considered as a generalization of the quaternions!

# Definition Clifford Algebra

- $V$: finite dimensional vector space over $\mathbb{K}$ (in practice $\mathbb{R}$ or $\mathbb{C}$)
- $\beta : V \times V \longrightarrow \mathbb{K}$ bilinear form
- A Clifford algebra $\mathrm{Cliff}\,(V, \beta)$ is an associative algebra with unit 1 over $\mathbb{K}$ and a linear injective map $\gamma : V \longrightarrow \mathrm{Cliff}\,(V, \beta)$ such that $\{\gamma(x), \gamma(y)\} = 2\beta(x, y)$, $1 \notin \gamma(V)$ and $\gamma(V)$ generates $\mathrm{Cliff}\,(V, \beta)$ as an algebra
- Given any associative algebra $\mathcal{A}$ with unit element 1 and a linear map $\varphi : V \longrightarrow \mathcal{A}$ such that $\{\varphi(x), \varphi(y)\} = 2\beta(x, y)$ there exists an associative algebra homomorphism $\widetilde{\varphi} : \mathrm{Cliff}\,(V, \beta) \longrightarrow \mathcal{A}$ such that $\varphi = \widetilde{\varphi} \circ \gamma$

$$\begin{array}{ccc} V & \longrightarrow^{\gamma} & \mathrm{Cliff}\,(V, \beta) \\ & {}_{\varphi} \searrow & \downarrow^{\tilde{\varphi}} \\ & & \mathcal{A} \end{array} \tag{10}$$

# Some Properties and Examples

- We start with the tensor algebra $\mathcal{T}(V)$ and take the two-sided ideal $\mathcal{J}(V,\beta)$ generated by $x \otimes y - y \otimes x - 2\beta(x,y)$ for $x, y \in V$. We define $\mathrm{Cliff}(V,\beta) = \mathcal{T}(V)/\mathcal{J}(V,\beta)$
- We might identify $V$ with $\gamma(V)$ and $\mathbb{K}$ can be identified with $\mathrm{span}(1)$.
- If $\dim V = n$ then as a vector space $\dim \mathrm{Cliff}\,(V,\beta) = 2^n$
- Over $\mathbb{R}$ define $q_{n.m}(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i=n+1}^{m+n} x_i^2$ . Then

$$\begin{aligned} \mathrm{Cl}(0,1) &\simeq \mathbb{C} \\ \mathrm{Cl}\,(0,2) &\simeq \mathbb{H} \end{aligned} \tag{11}$$

- Remark: $\mathrm{Cl}(0,3)$ are not the octonions since they are not even associative.

# Superalgebra

- If we define $\Pi : V \longrightarrow V$ by $\Pi(v) = -v$ then by the universal property $\Pi$ extends to an automorphism $\Pi : \mathrm{Cliff}\,(V, \beta) \longrightarrow \mathrm{Cliff}\,(V, \beta)$
- We define the even part $\mathrm{Cliff}^{+} = \{v \in \mathrm{Cliff} : \Pi(v) = v\}$ and the odd part $\mathrm{Cliff}^{-} = \{v \in \mathrm{Cliff} : \Pi(v) = -v\}$. We have that

$$\begin{aligned} \mathrm{Cliff}^{+}\mathrm{Cliff}^{+} &\subseteq \mathrm{Cliff}^{+} \\ \mathrm{Cliff}^{-}\mathrm{Cliff}^{-} &\subseteq \mathrm{Cliff}^{+} \\ \mathrm{Cliff}^{+}\mathrm{Cliff}^{-} &\subseteq \mathrm{Cliff}^{-} \\ \mathrm{Cliff}^{-}\mathrm{Cliff}^{+} &\subseteq \mathrm{Cliff}^{+} \end{aligned} \tag{12}$$

- Moreover,

$$\mathrm{Cliff} = \mathrm{Cliff}^{+} \oplus \mathrm{Cliff}^{-} \tag{13}$$

This turns the Clifford Algebra into a Superalgebra.

# Clifford Algebras as a Quantization Procedure

- If we take $q(x) = 0$ then the Clifford algebra is the exterior algebra $\bigwedge V$
- In this way, if we take a non-degenerate symmetric bilinear form $b(\cdot,\cdot)$ by a parameter $t$ gives a Clifford Algebra $\mathrm{Cliff}(V, tb(\cdot,\cdot))$ which can be considered as a deformation of the exterior algebra $\bigwedge V$
- As a superalgebra, the exterior algebra $\Lambda V$ is supercommutative while the Clifford Algebra is not. Therefore, the Clifford Algebra is a noncomutative version of the exterior algebra in this other sense

# Bott Periodicity

- For $n \geq 0$ we have the following isomorphism of complex associative algebras

$$\mathrm{Cliff}\left(\mathbb{C}^{n+2}\right) \simeq \mathrm{Cliff}\left(\mathbb{C}^{n}\right) \otimes \mathrm{Mat}_{2\times 2}\left(\mathbb{C}\right) \tag{14}$$

- To see this write $\mathbb{C}^{n+2} = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}^n$ where $e_1, \cdots, e_{n+2}$ generate $\mathrm{Cliff}\left(\mathbb{C}^{n+2}\right)$ and $e_1^*, \cdots, e_n^*$ generate $\mathrm{Cliff}\left(\mathbb{C}^{n}\right)$

$$\phi : \mathbb{C}^{n+2} \longrightarrow \mathrm{Cliff}\left(\mathbb{C}^{n}\right) \otimes \mathrm{Mat}_{2\times 2}\left(\mathbb{C}\right)$$

$$\phi\left(e_1\right) = 1 \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \phi\left(e_2\right) = 1 \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{15}$$

$$\phi(e_j) = (ie_{j-2}^*) \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad 3 \leq j \leq n+2$$

# Classification of Clifford Algebras over $\mathbb{C}$

- From Bott periodicity and the fact that

$$\begin{cases} \mathrm{Cliff}\left(\mathbb{C}^0\right) = \mathrm{Cliff}\left(1\right) \simeq \mathbb{C} \\ \mathrm{Cliff}\left(\mathbb{C}\right) \simeq \mathbb{C} \oplus \mathbb{C} \end{cases} \tag{16}$$

  it follows that for $k \geq 0$

$$\begin{cases} \mathrm{Cliff}\left(\mathbb{C}^{2k}\right) \simeq \mathbb{C} \bigotimes_{i=1}^{k} \mathrm{Mat}_{2\times 2}\left(\mathbb{C}\right) \simeq \mathrm{End}\left(\mathbb{C}^{2^k}\right) \\ \mathrm{Cliff}\left(\mathbb{C}^{2k+1}\right) \simeq \left(\mathbb{C} \oplus \mathbb{C}\right) \bigotimes_{i=1}^{k} \mathrm{Mat}_{2\times 2}\left(\mathbb{C}\right) \simeq \mathrm{End}\left(\mathbb{C}^{2^k}\right) \oplus \mathrm{End}\left(\mathbb{C}^{2^k}\right) \end{cases} \tag{17}$$

- For $n = 2k, 2k+1$

$$\triangle_n = \mathbb{C}^{2^k} \tag{18}$$

  is called the spinor space

# The Spin Group

- Over $\mathbb{R}^n$ consider $q(x) = -x_1^2 - \cdots - x_n^2$. Observe that $x^2 = -\|x\|^2$.
- The Pin Group $\mathrm{Pin}(n)$ consists of the group generated under the Clifford Multiplication by all vectors $x \in S^{n-1}$
- The Spin Group is

$$\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap \mathrm{Cliff}^{\,+} \tag{19}$$

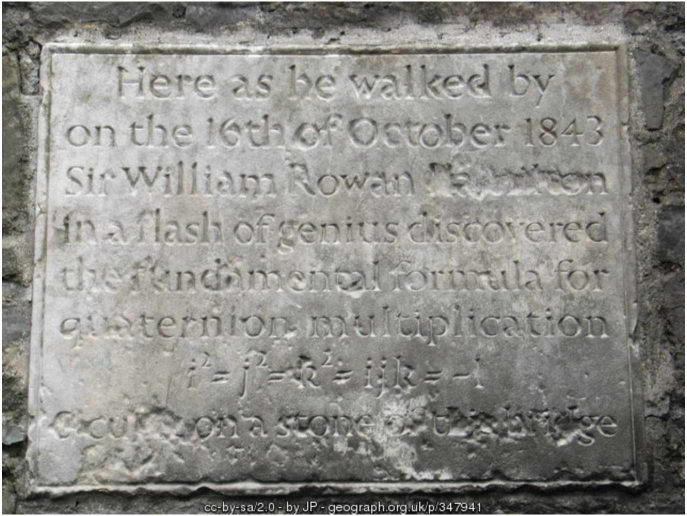
- It can be shown that

$$\begin{gathered} \lambda : \mathrm{Pin}(n) \longrightarrow O(n) \\ \lambda(x)y = xyx^T \end{gathered} \tag{20}$$

is a group homomorphism which is a dobule cover of $O(n)$
- For example,

$$\begin{gathered} \mathrm{Spin}(2) = SO(2) \\ \mathrm{Spin}(3) = SU(2) \end{gathered} \tag{21}$$

# Thank you!


Here as he walked by
on the 16th of October 1843
Sir William Rowan
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
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