Naturality of the contact invariant in monopole Floer homology under strong symplectic cobordisms

Mariano Echeverria

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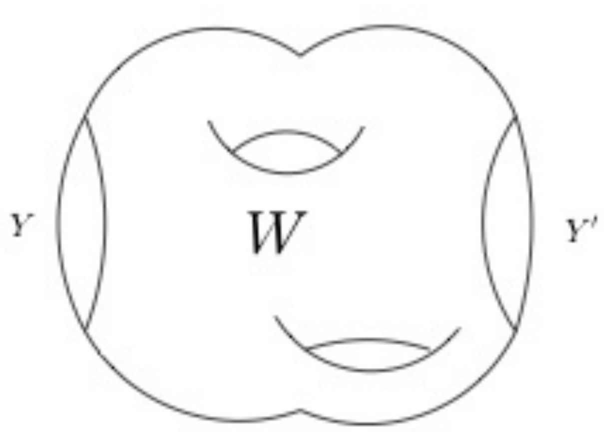
- Each group admits a decomposition

$$HM_\bullet(Y, \mathfrak{s}) = \bigoplus_{[\xi]} HM_\bullet(Y, \mathfrak{s}, [\xi])$$

  where $[\xi]$ denotes a homotopy class of oriented plane fields.

TFQT Features of Monopole Floer Homology

$$(W, \mathfrak{s}_W) : (Y, \mathfrak{s}_Y) \to (Y', \mathfrak{s}_{Y'})$$



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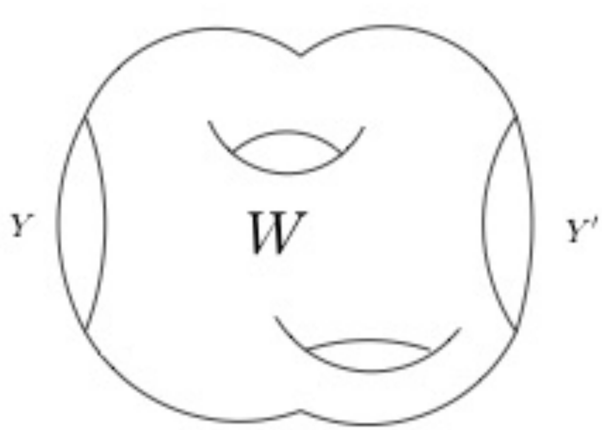

Induces maps for each of the flavors

$$HM^{\bullet}(W, \mathfrak{s}_W) : HM^{\bullet}(Y', \mathfrak{s}_{Y'}) \to HM^{\bullet}(Y, \mathfrak{s}_Y)$$

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The Contact Invariant and the Naturality Problem

Given (Y, ξ) , KMOS defined the **contact invariant**

$$\mathbf{c}(\xi) \in \widehat{HM}^{\bullet}(Y, \mathfrak{s}_{\xi}) \simeq \widecheck{HM}_{\bullet}(-Y, \mathfrak{s}_{\xi})$$

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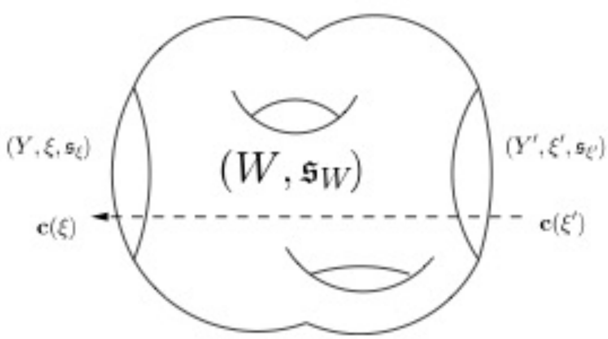

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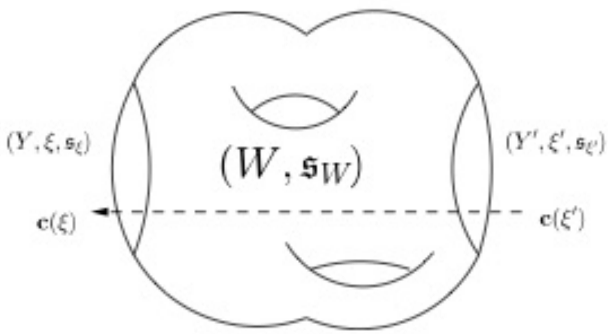

Naturality Problem: For which $(W, \mathfrak{s}_W)$ is it true that

$$\widehat{HM}^{\bullet}(W, \mathfrak{s}_W)\mathbf{c}(\xi') = \mathbf{c}(\xi)$$

Naturality under Strong Symplectic Cobordisms

Theorem (E. 2018)
Let $(W,\omega) : (Y,\xi) \to (Y',\xi')$ be a strong symplectic cobordism between two contact manifolds (Y,ξ) and (Y',ξ'). Then

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- The result is not known for Heegaard Floer in such generality.
- Michael Hutchings is currently writing the corresponding result for Embedded Contact Homology.

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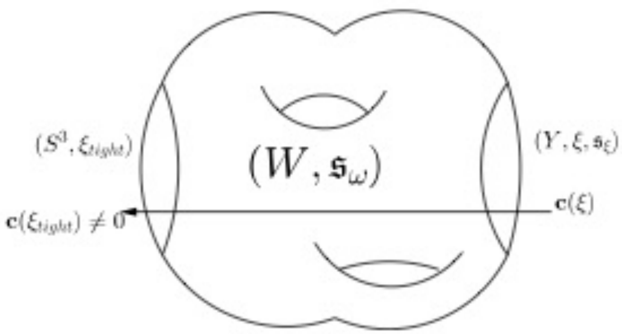

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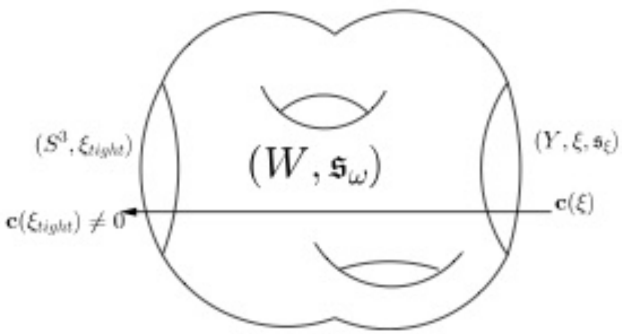

- Ghiggini gave examples of weak fillings where the contact invariant vanishes, so **the naturality result cannot be naively extended to the case of weak symplectic cobordisms**.

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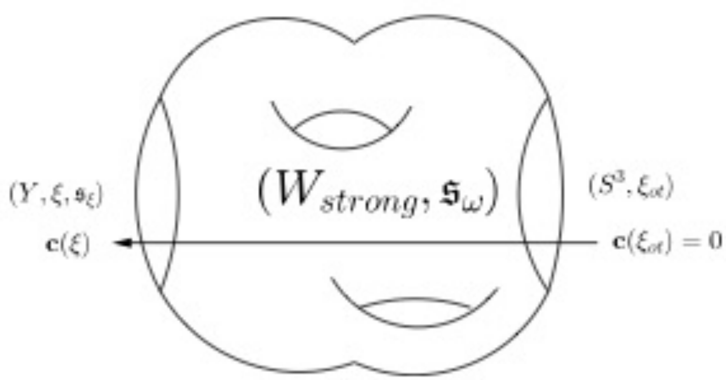
- If (Y, ξ) is overtwisted by Etnyre-Honda we can find a Stein cob.

$$(W_{strong}, \mathfrak{s}_\omega) : (Y, \xi, \mathfrak{s}_\xi) \to (S^3, \xi_{ot})$$



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i) (Ozsváth, Stipsciz and Szabó) (E.) The reduced part of the contact invariant vanishes, i.e, $[\mathbf{c}(\xi)]_{red} = 0$.

ii) (Etnyre) (E.) Any strong filling of (Y, ξ) must be negative definite.

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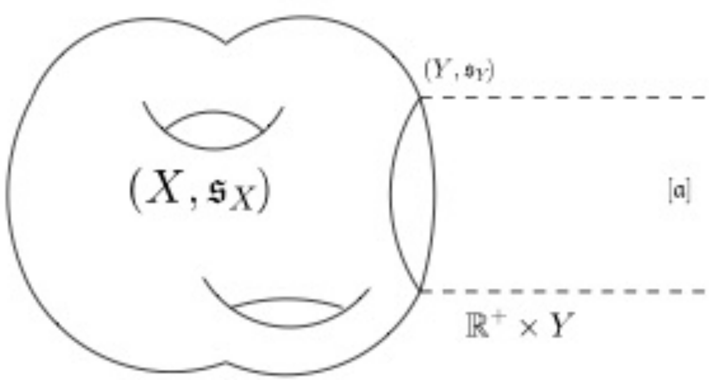
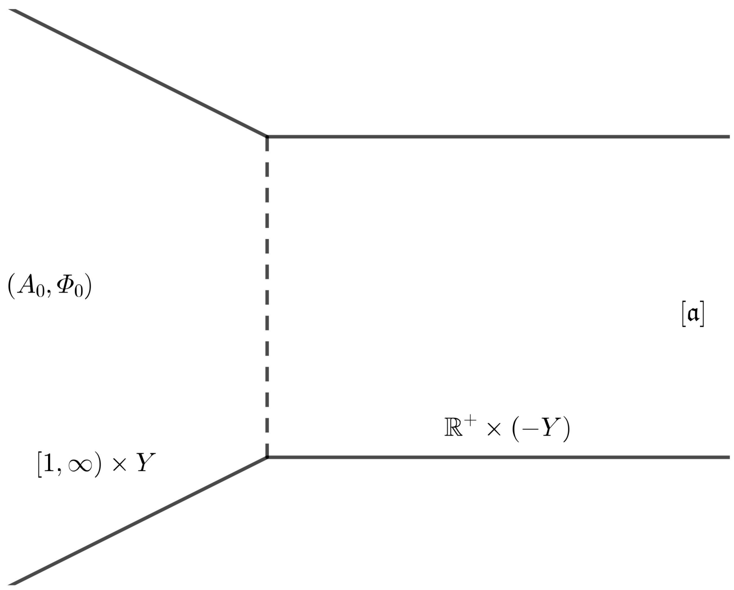
$$\varphi_{X,\mathfrak{s}_X} = \sum_{[\mathfrak{a}]} n_{[\mathfrak{a}]}[\mathfrak{a}] = \sum_{[\mathfrak{a}]} \left(\sum \#\mathcal{M}_0(X^*; [\mathfrak{a}]) \right) [\mathfrak{a}]$$



Definition of the Contact Invariant

The contact invariant $\mathbf{c}(\xi)$ of (Y, ξ) is the relative invariant associated to the symplectization of ξ



Building the Cobordism Maps

$$HM_\bullet(W, \mathfrak{s}_W) : HM_\bullet(Y, \mathfrak{s}) \to HM_\bullet(Y', \mathfrak{s}')$$
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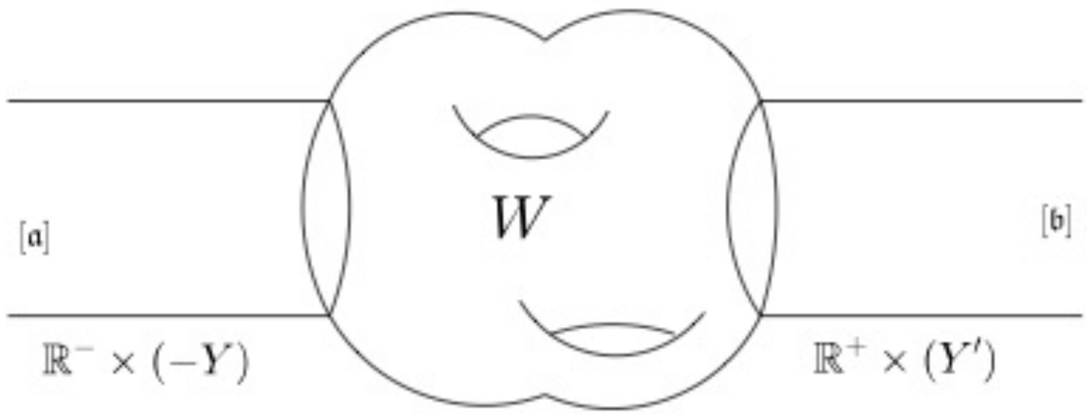

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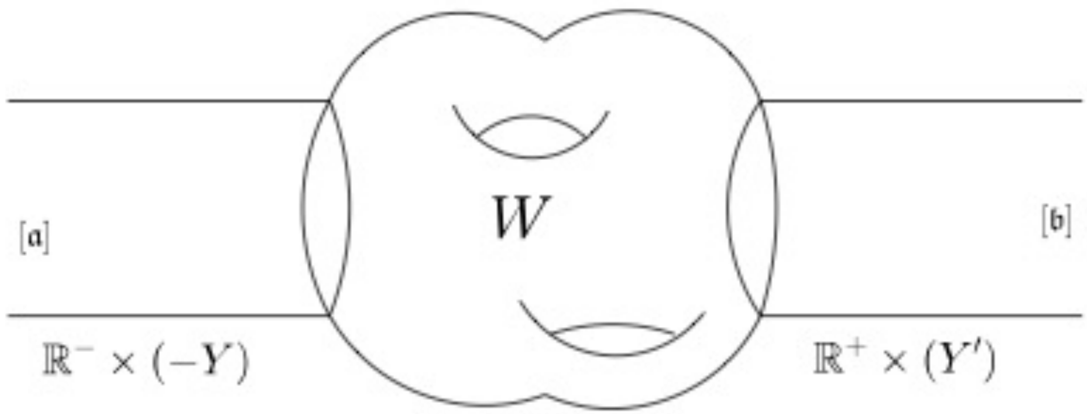
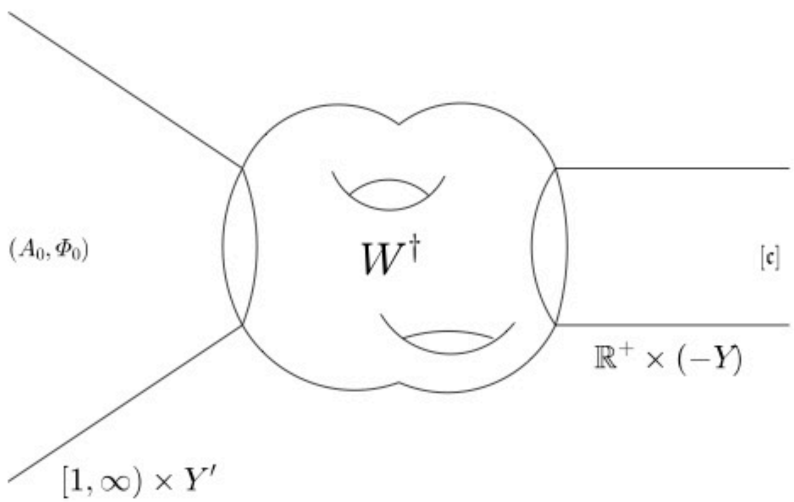


$$n_{[\mathfrak{a}],[\mathfrak{b}]} = \sum \#\mathcal{M}_0([\mathfrak{a}], W^*, [\mathfrak{b}])$$

Showing the Naturality Result: "hybrid invariant"

Use the conical end coming from (Y', ξ') together with the cylindrical end coming from (Y, ξ) to produce a "hybrid invariant"

$$\mathbf{c}(\xi', Y) \in \widehat{HM}^{\bullet}(Y, \mathfrak{s}_\xi) \simeq \widetilde{HM}_{\bullet}(-Y, \mathfrak{s}_\xi)$$



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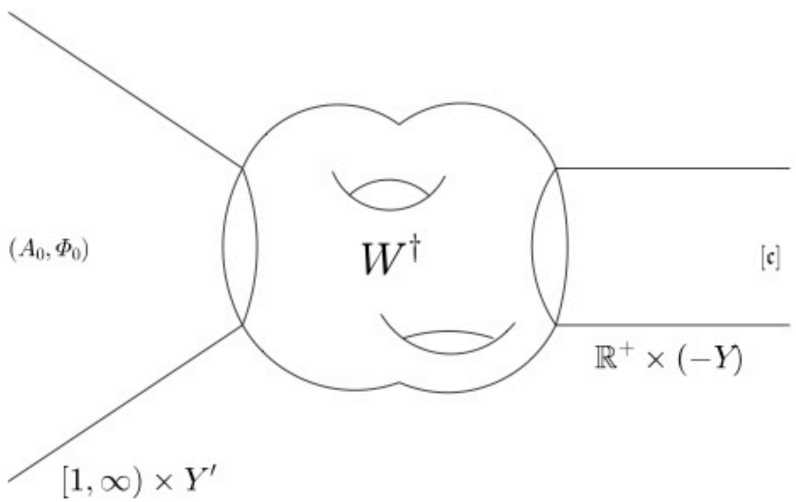

$$\widehat{HM}^{\bullet}(W, \mathfrak{s}_{\omega})\mathbf{c}(\xi') = \mathbf{c}(\xi', Y) = \mathbf{c}(\xi)$$

Stretching the Neck Argument

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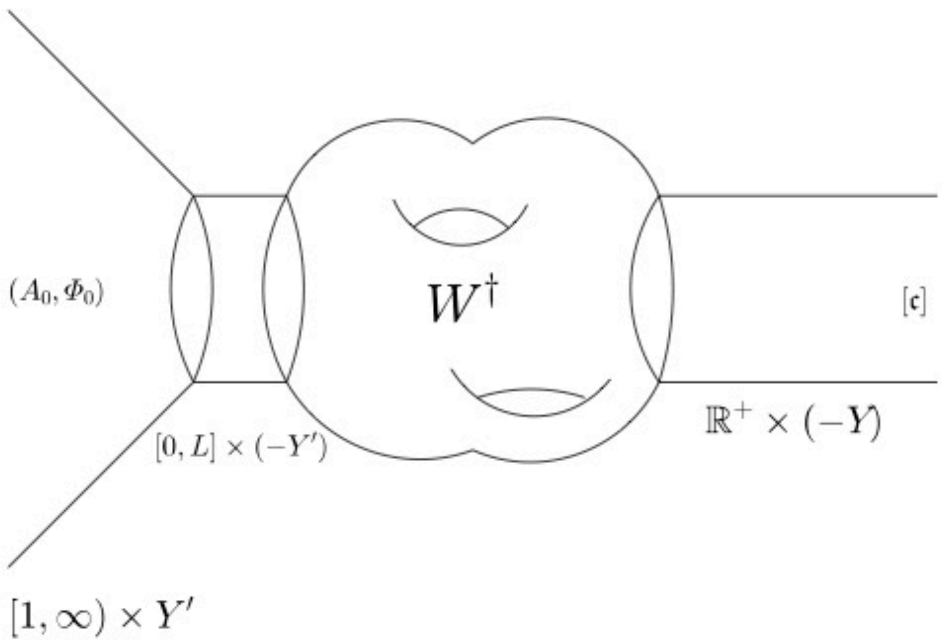

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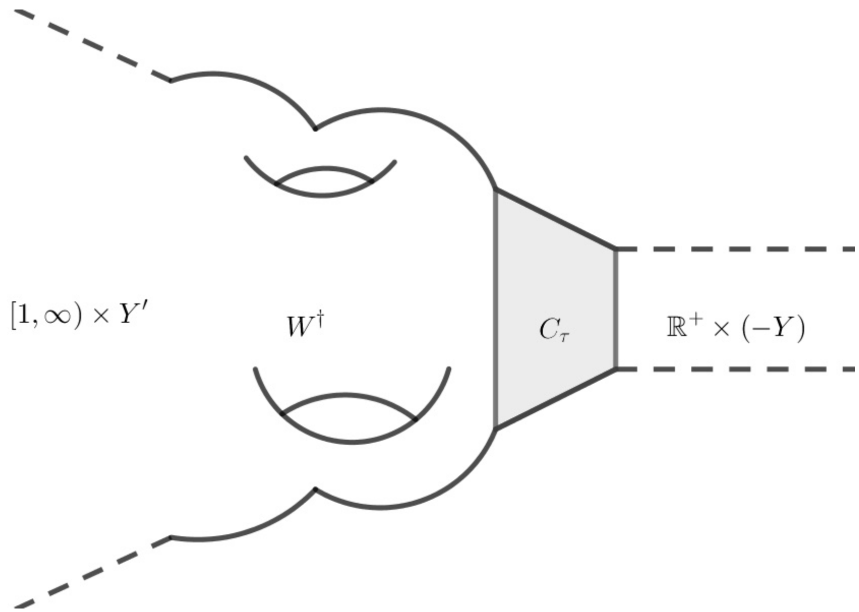

Thank You!

[image taken from Patrick Massot's website]