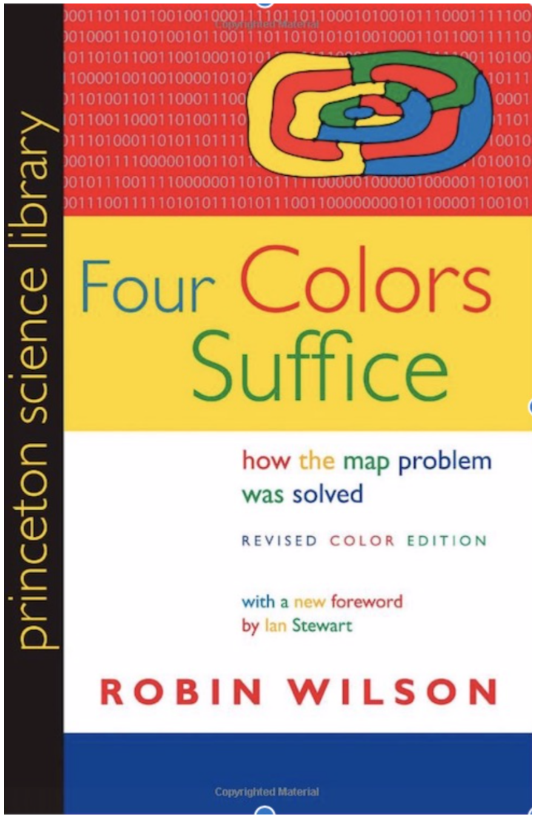
The 4–color theorem and Gauge theory

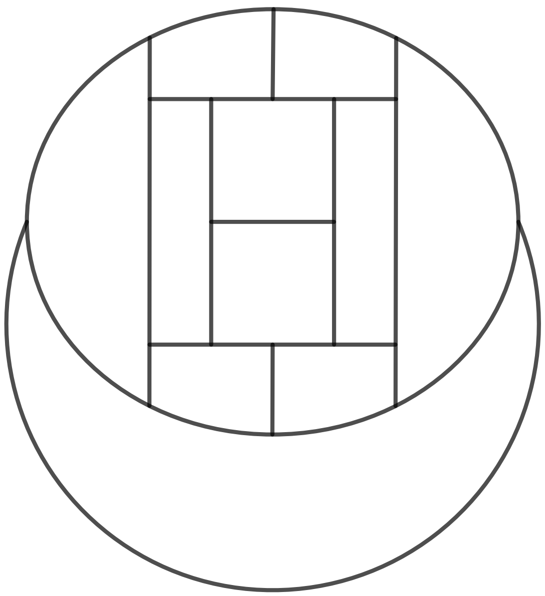
Mariano Echeverria

Tait's Equivalence

Theorem. *The four-color theorem is equivalent to the statement: every bridgeless, trivalent, planar graph K admits a Tait coloring.*



Appel and Haken: Four colors suffice!



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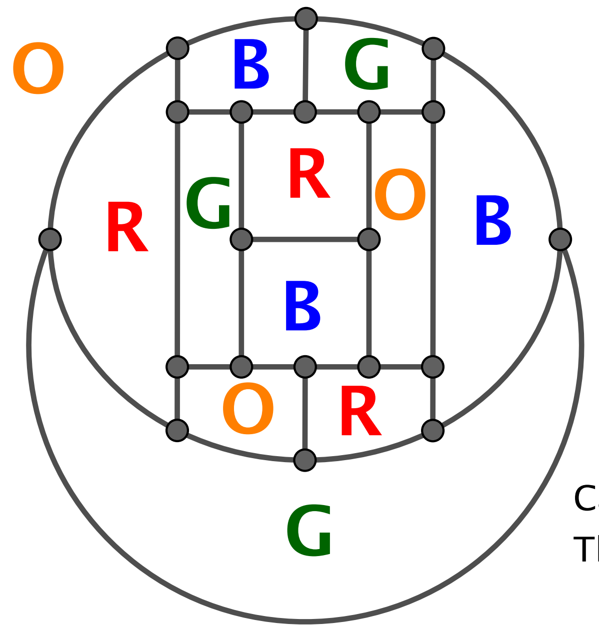
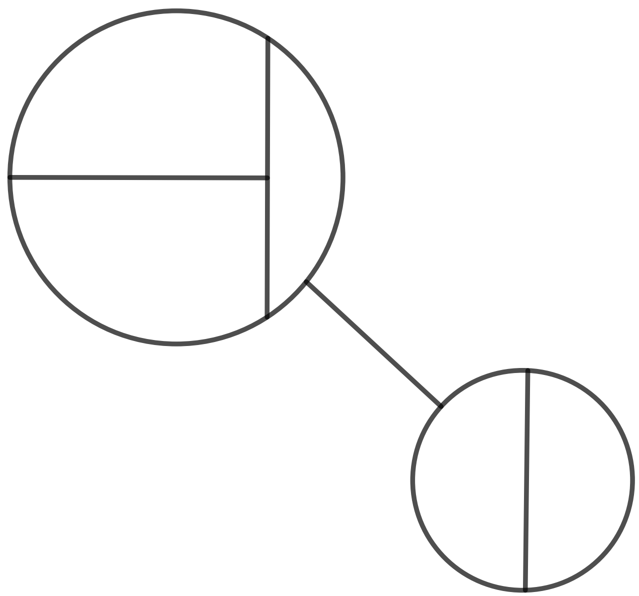
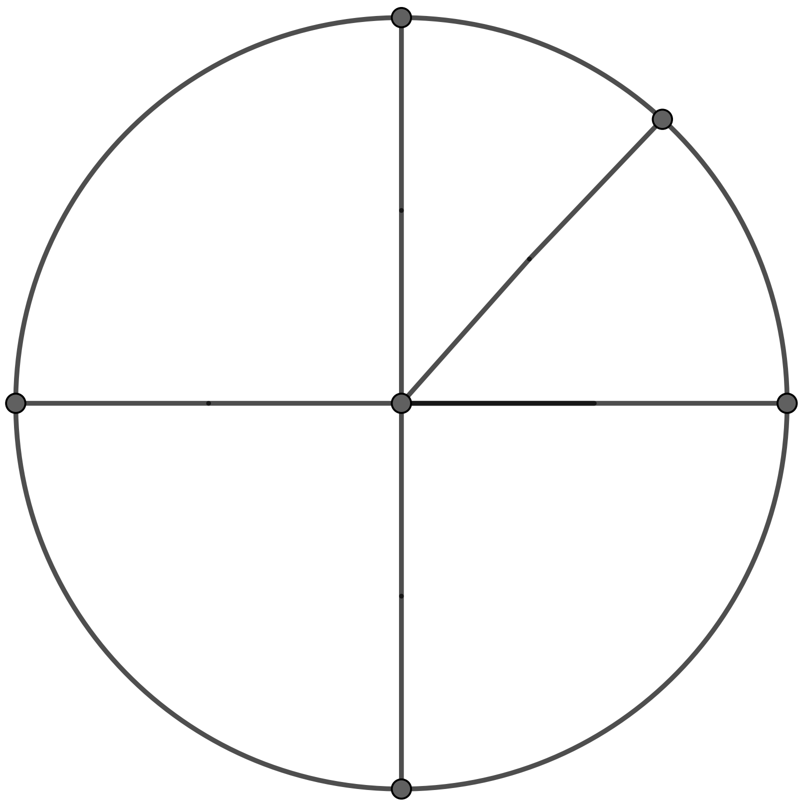


Can color it on the 2-sphere
The graph is trivalent

Avoiding Bridges!!!



Making the graph trivalent



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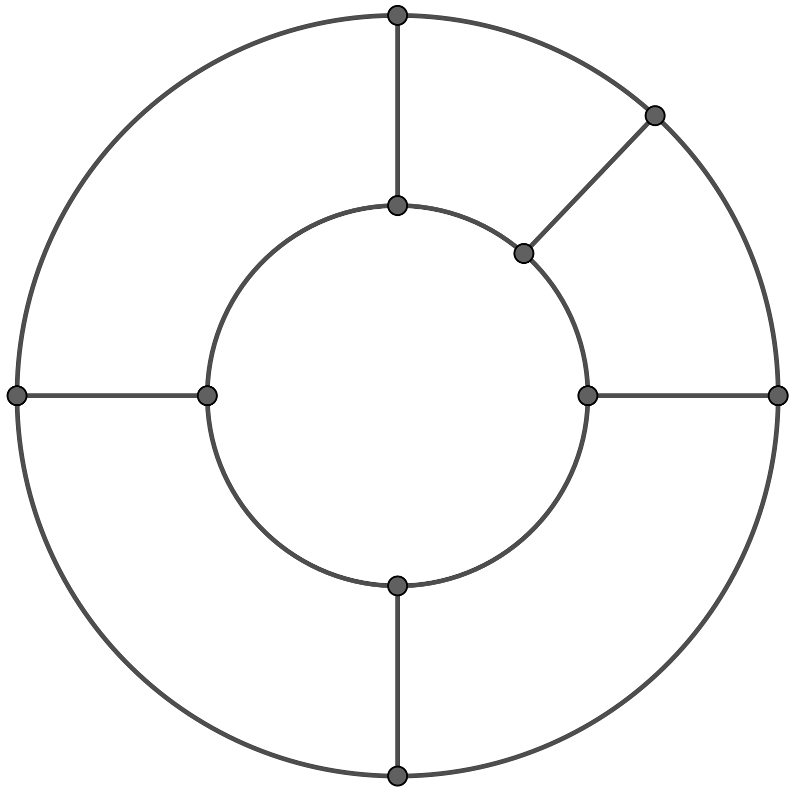

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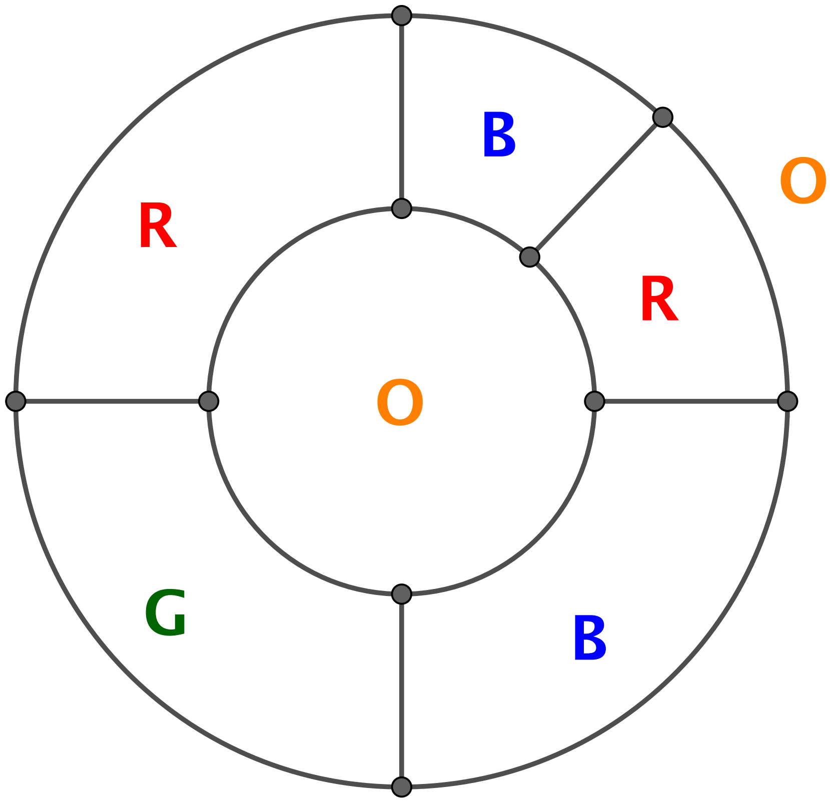

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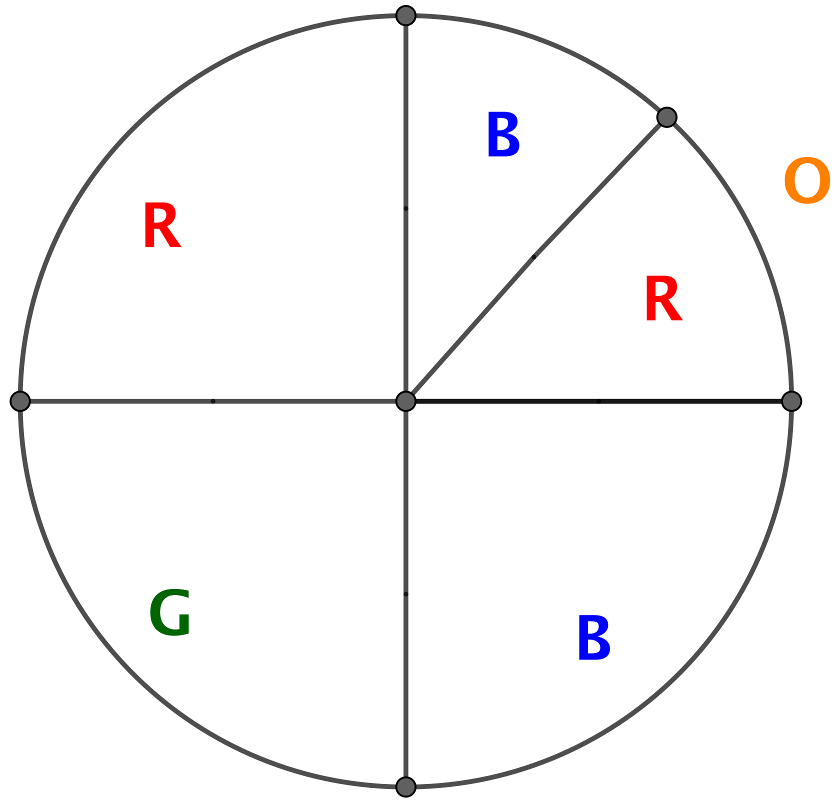
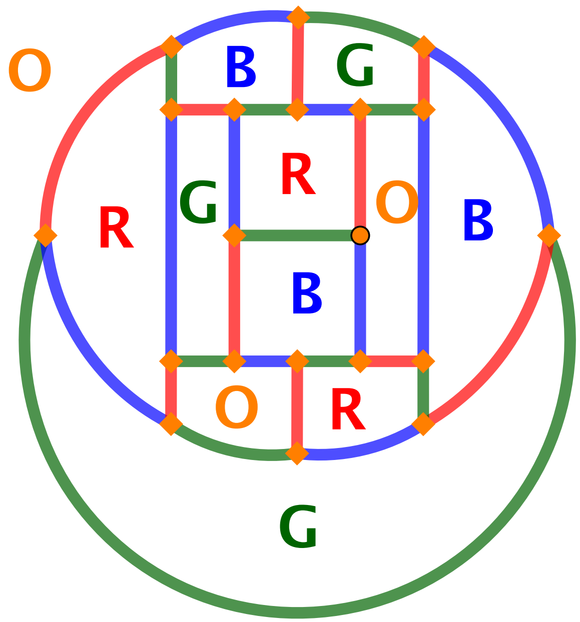


Tait Colorings and the Klein 4-Group V_4



O=diag(1,1,1)

R=diag(1,−1,−1)

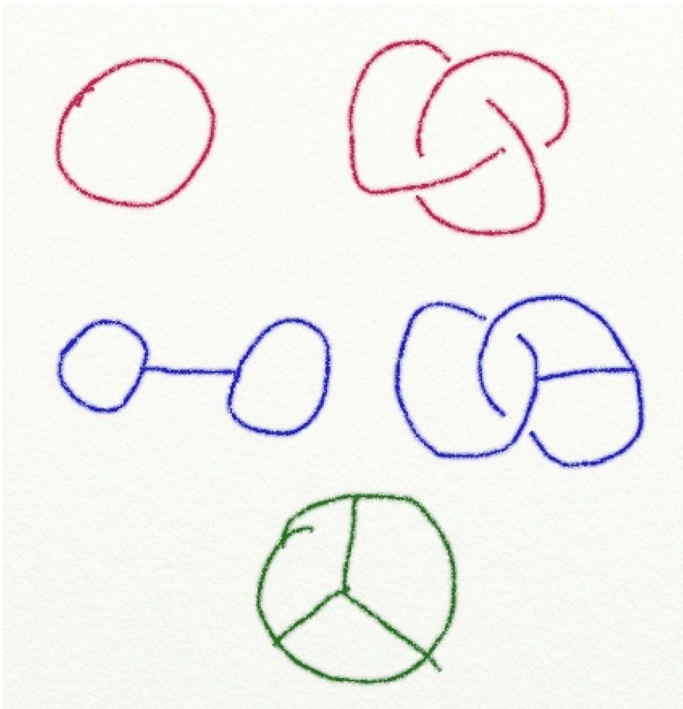
G=diag(−1,1,−1)

B=diag(−1,−1,1)

{O,R,G,B} satisy the group law of the Klein 4−group

Webs

We want to consider the embedding of a graph K in $\mathbb{R}^3$ (or S^3)



$p \in K$ has a neighborhood $\mathbf{Y}$ or $-$

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- spatial web $K \implies J^{\#}(K)$ a finite dimensional vector space over $\mathbb{F}_2$

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Conjecture. *If* K *is* ***planar,*** *i.e,* $K \subset \mathbb{R}^2 \subset \mathbb{R}^3$*, then* $\dim J^{\#}(K) =$ *Tait*(K)

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Remark. A bridgeless, planar, trivalent web K has $J^{\#}(K) \neq 0$ and if the conjecture is true, then Tait$(K) \neq 0$, **implying the 4-color theorem.**

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Remark. • A bridgeless, planar, trivalent web K has $J^{\#}(K) \neq 0$ and if the conjecture is true, then Tait$(K) \neq 0$, **implying the 4-color theorem.**
• The non-vanishing result uses Gabai's theory of **sutured manifolds**

Some Results and Questions

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Some Results and Questions

- One inequality of the conjecture is true, namely, if K is planar then $\dim J^{\#}(K) \geq \mathsf{Tait}(K)$
- How is $J^{\#}(K)$ related to Tait(K)?
- What properties does $J^{\#}(K)$ satisfy?

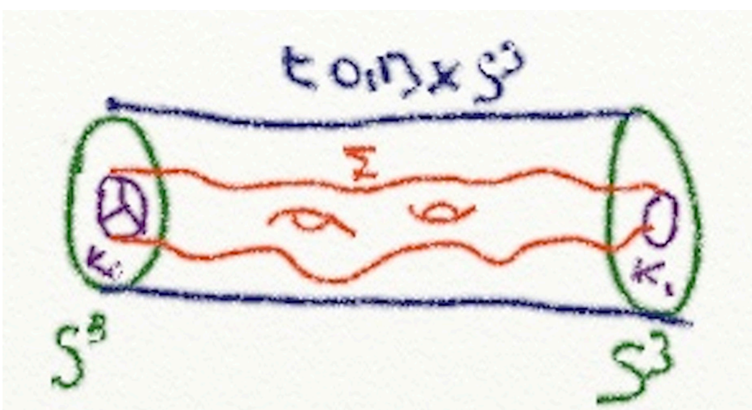
Maps induced by foam cobordisms

Foam cobordisms

$$([0,1] \times S^3, \Sigma) : (S^3, K_0) \to (S^3, K_1)$$

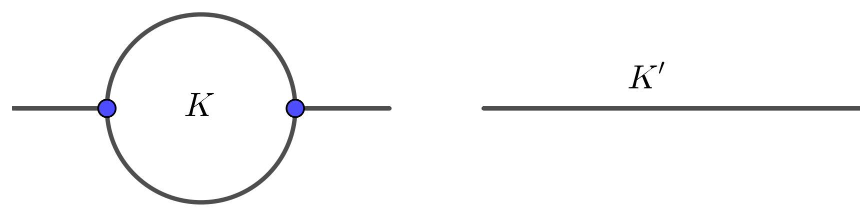
induce **linear transformations**

$$J^{\#}(\Sigma) : J^{\#}(K_0) \to J^{\#}(K_1)$$



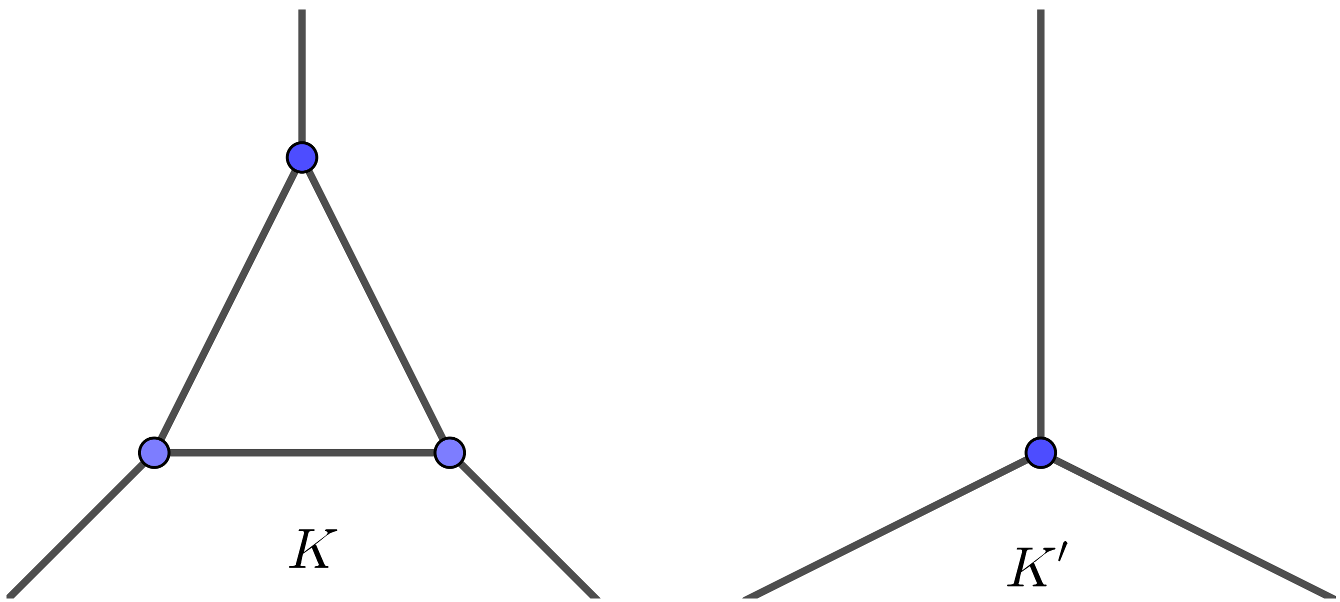
Bigon Relations

$$\dim J^{\#}(K) = 2 \dim J^{\#}(K')$$



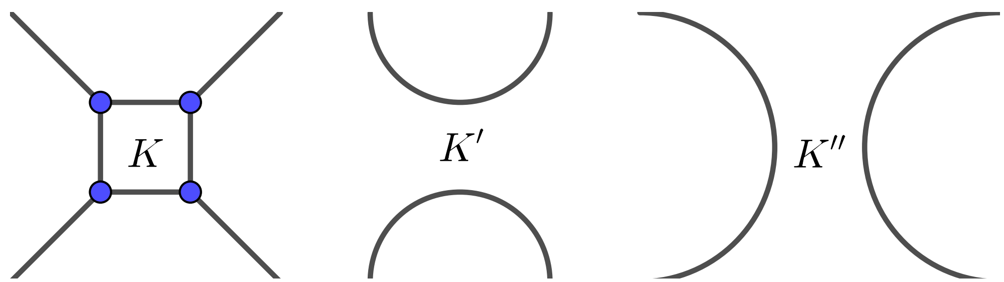
Triangle Relations

$$J^{\#}(K) \simeq J^{\#}(K')$$



Square Relations

$$J^{\#}(K) \simeq J^{\#}(K') \oplus J^{\#}(K'')$$

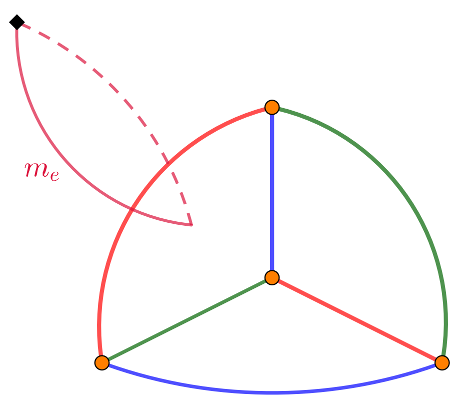


Tait Colorings and $\pi_1(S^3\backslash K)$

Tait coloring of $K \iff$ (group) homomorphism

$$\rho : \pi_1(S^3\backslash K) \to V_4$$

sending the m_e to elements of **order 2 (strictly)**



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Since V_4 is **abelian** $\rho : \pi_1(S^3\backslash K) \to V_4$ is the same as

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If e_1, e_2, e_3 meet at a vertex then in $H_1(S^3 \backslash K)$

$$[m_{e_1}] + [m_{e_2}] + [m_{e_3}] = 0$$

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so

$$\rho(m_{e_1})\rho(m_{e_2})\rho(m_{e_3}) = 1_{3\times 3}$$

i.e, $\rho(m_{e_1}), \rho(m_{e_2}), \rho(m_{e_3})$ must be given different colors!!!

Tait(unknot)

- $\pi_1(S^3 \backslash \text{unknot}) = \mathbb{Z}$ so

$$\rho : \mathbb{Z} \to V_4 = \{1, R, G, B\}$$

is completely specified by

$$\rho(1) = \rho(m)$$

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$$\rho(1) = \rho(m)$$

- Since $\rho(m)$ cannot be the identity there are 3 possible choices so

$$\text{Tait}(unknot) = 3$$

Construction of $J^{\#}(K)$

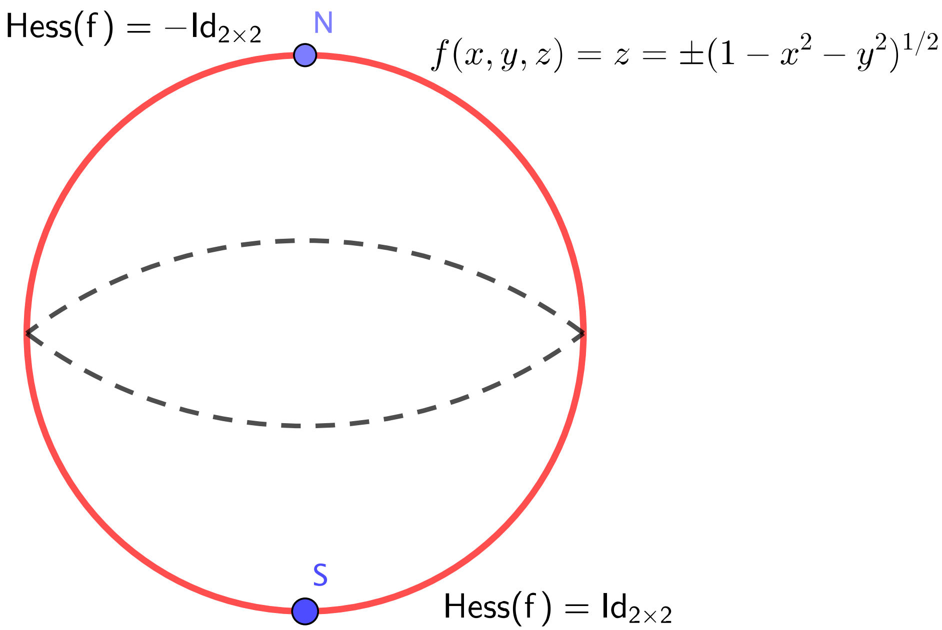
$J^{\#}(K)$ is constructed as the **homology groups** of a chain complex

$$(C_\bullet, \partial_\bullet) \implies J^{\#}(K) = \frac{\ker \partial_\bullet}{\text{im}\partial_\bullet}$$

built from the **character variety** of K

$$\mathcal{R}^{\#}(K) = \{\rho : \pi_1(S^3 \backslash K) \to SO(3) \mid \rho(m_e) \text{ has order } 2 \text{ for all edges } e\}$$

Morse Homology



Morse Homology of f

$$0 \rightarrow^{\partial_3} \underbrace{C_2}_{\mathbb{F}_2[N]} \rightarrow^{\partial_2} \underbrace{C_1}_{0} \rightarrow^{\partial_1} \underbrace{C_0}_{\mathbb{F}_2[S]} \rightarrow^{\partial_0} 0$$

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$$H_2(f, S^2; \mathbb{F}_2) = \frac{\ker \partial_2}{\mathrm{im} \partial_3} = \ker \partial_2 = \mathbb{F}_2$$

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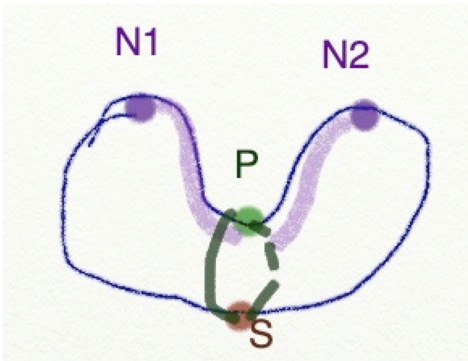
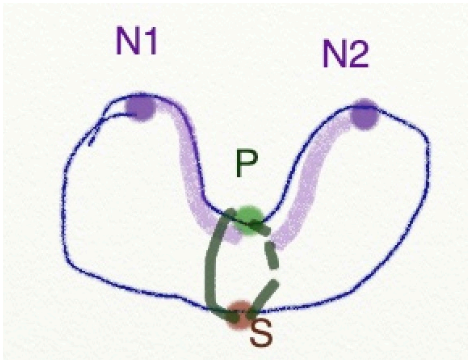
$$\boxed{H_\bullet(f, S^2; \mathbb{F}_2) \simeq H_\bullet(S^2; \mathbb{F}_2)}$$

Morse Homology of g



$$0 \to^{\partial_3} \underbrace{C_2}_{\mathbb{F}_2[N_1] \oplus \mathbb{F}_2[N_2]} \to^{\partial_2} \underbrace{C_1}_{\mathbb{F}_2[P]} \to^{\partial_1} \underbrace{C_0}_{\mathbb{F}_2[S]} \to^{\partial_0} 0$$

Morse Homology of g (mod 2)



$$\partial_2[N_1] = [P] \quad \partial_1[P] = 2[S] = 0 \quad \partial_0[S] = 0$$

$$\partial_2[N_2] = [P]$$

Morse Homology of g (mod 2)

$$\partial_2 N_1 = P \qquad \partial_1 P = 2S = 0 \qquad \partial_0 S = 0$$

$$\partial_2 N_2 = P$$

$$H_2(g, S^2; \mathbb{F}_2) = \frac{\ker \partial_2}{\mathrm{im}\partial_3} = \mathbb{F}_2$$

$$H_1(g, S^2; \mathbb{F}_2) = \frac{\ker \partial_1}{\mathrm{im}\partial_2} = 0$$

$$H_0(g, S^2; \mathbb{F}_2) = \frac{\ker \partial_0}{\mathrm{im}\partial_1} = \mathbb{F}_2$$

Same answer as before!

Ingredients for a Morse Homology

- A real valued function $f : M \to \mathbb{R}$

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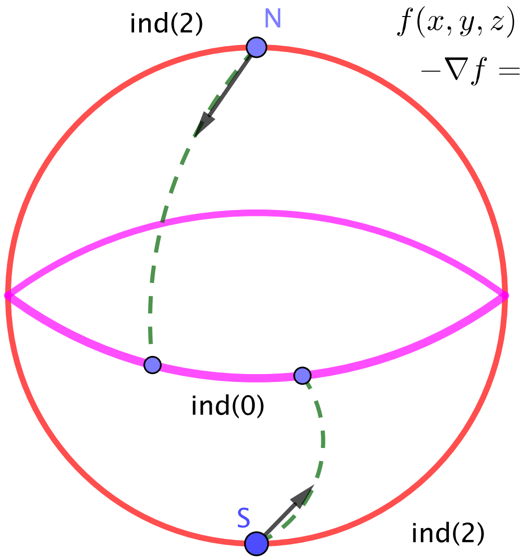
- The differential ∂ is build from flow lines $\gamma(t)$ of the vector field $-$grad$f$: $\frac{d\gamma}{dt} = -\text{grad} f(\gamma(t))$

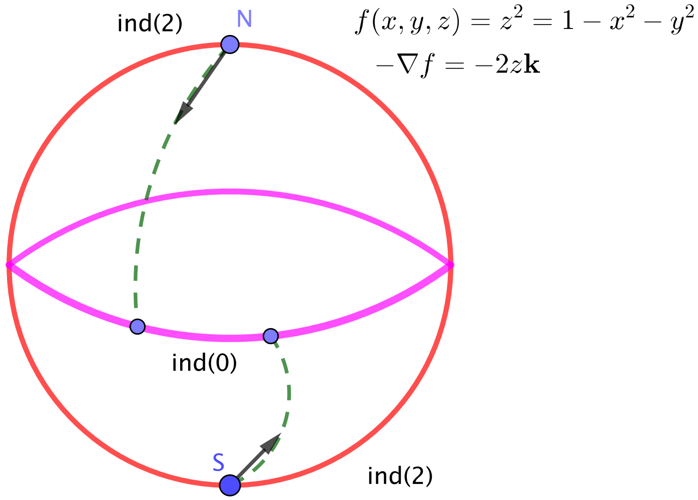
Morse-Bott Situation



$$f(x,y,z) = z^2 = 1 - x^2 - y^2$$

$$-\nabla f = -2z\mathbf{k}$$

Morse-Bott Situation



Crit(f) now consists of submanifolds, and the homology of the manifold can still be recovered modifying the previous construction!

Idea behind $J^{\#}(K)$: Floer Homology

- On a 3-manifold Y (like $Y = S^3 \backslash K$), the representations

$$\rho : \pi_1(Y) \to SO(3)$$

can be interpreted as the critical points of a functional $f = CS$ which plays the role of a Morse function!

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- Then one can try to mimic the construction of the Morse Homology groups using CS and the "gradient-flow lines" determined by $-\text{grad}CS$
- $H_\bullet(CS, Y; \mathbb{F}_2)$ no longer gives $H_\bullet(Y; \mathbb{F}_2)$, but rather new and interesting topological invariants of Y!

$U(1)$ Chern-Simons Theory

$$CS : \{\text{vector fields on } Y\} \to \mathbb{R}$$
$$\mathbf{A} \to \int_Y \mathbf{A} \cdot (\nabla \times \mathbf{A}) dvol_Y$$

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Modulo integration by parts arguments

$$\begin{aligned}
&\mathcal{D}_{\mathbf{a}} CS(A) \\
=&\lim_{t \to 0} \frac{CS(\mathbf{A} + t\mathbf{a}) - CS(A)}{t} \\
=&2 \int_Y \mathbf{a} \cdot (\nabla \times \mathbf{A}) dvol_{Y}
\end{aligned}$$

$U(1)$ Chern-Simons Theory

$$\mathcal{D}_{\mathbf{a}} CS(A) = 0 \quad \forall \mathbf{a} \iff \nabla \times \mathbf{A} = \mathbf{0}$$

so

$$\mathsf{Crit}(CS) = \{\mathbf{A} \in \mathcal{X}(Y) \mid \nabla \times \mathbf{A} = \mathbf{0}\}$$

Holonomy Correspondence for $U(1)$

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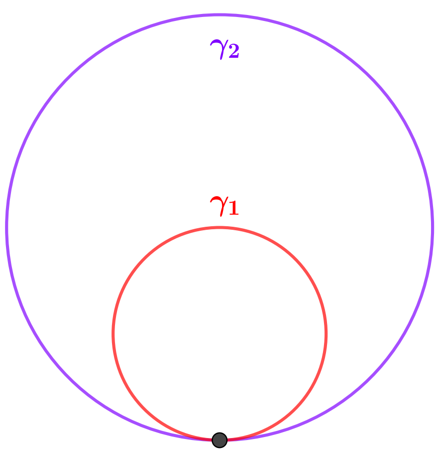
$$\mathrm{Hol}_{\mathbf{A}} : \pi_1(Y, y_0) \to U(1)$$

$$\gamma \to e^{i \int_\gamma \mathbf{A} \cdot d\mathbf{r}}$$



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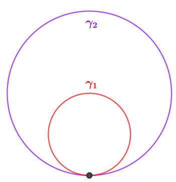

That this is independent of the representative of $[\gamma] \in \pi_1(Y, y)$ uses Stokes theorem and the fact that $\nabla \times \mathbf{A} = \mathbf{0}$

Holonomy Correspondence for $U(1)$

- The **holonomy correspondence** says that Hol is reversible, in particular, every $\rho \in \hom(\pi_1(Y), U(1))$ can be written as

$$\rho = \mathsf{Hol}_{\mathbf{A}}$$

for some $\mathbf{A} \in \mathcal{X}(Y)$, which means that ρ "solves" the P.D.E

$$\nabla \times \mathbf{A} = \mathbf{0}$$

Back to the unknot $\circ$

$J^{\#}(\circ)$ is the homology with critical set

$$\mathcal{R}^{\#}(\circ)$$
$$=\{\rho : \pi_1(S^3 \backslash \circ) \to SO(3) \mid \rho(m_e) \text{ has order } 2\}$$

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=&\{\rho : \mathbb{Z} \to SO(3) \mid \rho(1) \text{ has exactly order } 2\}\\
=&\{\text{rotation along about an axis by 180 degrees}\}\\
=&\mathbb{S}^2/\mathbb{Z}_2\\
=&\mathbb{RP}^2
\end{aligned}
$$

this is **Morse-Bott!!!**

Back to the unknot $\circ$

In a nutshell, one can arrange things so that

$$J^{\#}(\circ)$$
$$= H_\bullet\left(\mathbb{RP}^2; \mathbb{Z}_2\right)$$

Back to the unknot $\circ$

In a nutshell, one can arrange things so that

$$\begin{aligned} & J^{\#}(\circ) \\ = & H_{\bullet}\left(\mathbb{RP}^2; \mathbb{Z}_2\right) \\ = & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \end{aligned}$$

Back to the unknot $\circ$

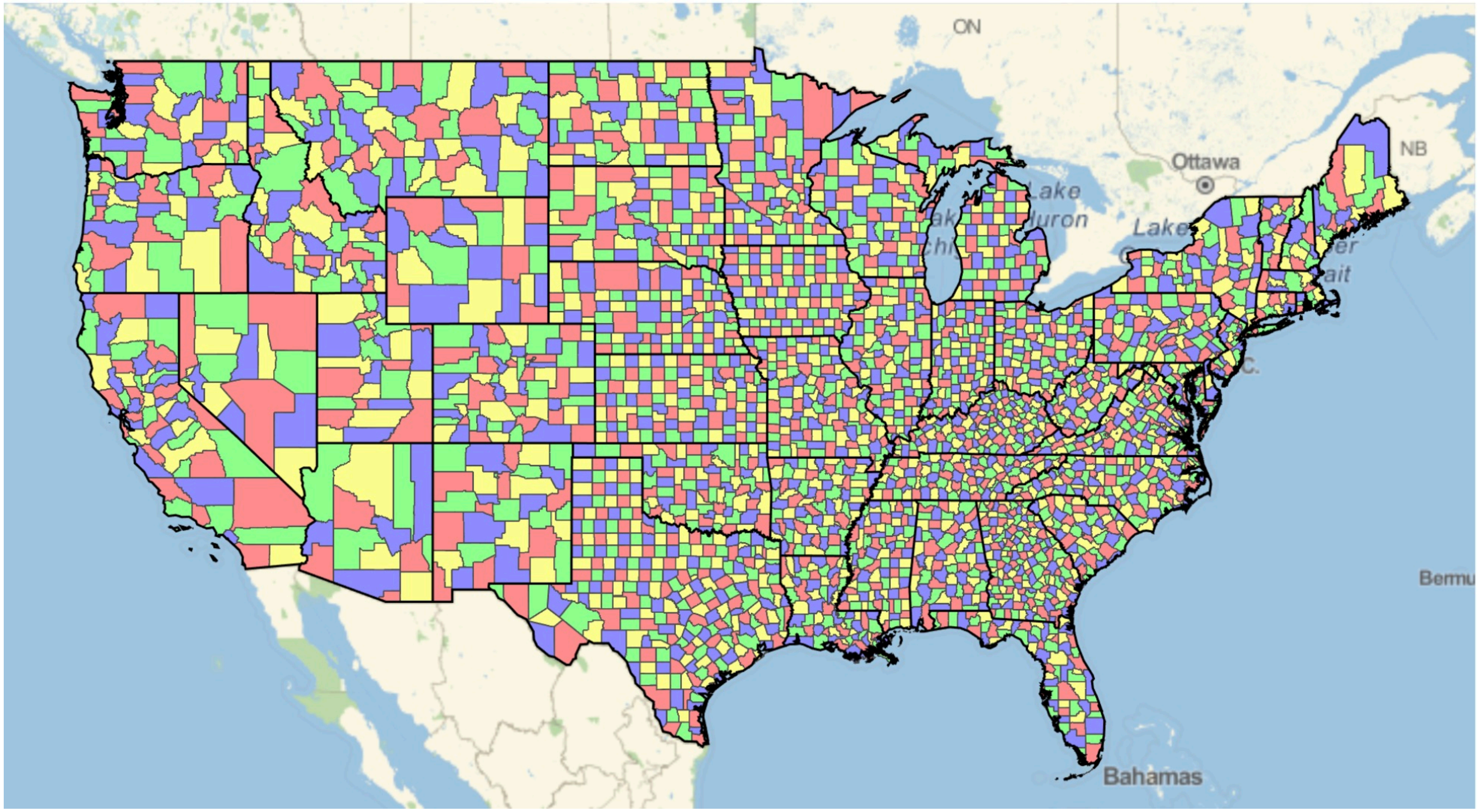
In a nutshell, one can arrange things so that

$$\begin{aligned} & J^{\#}(\circ) \\ =& H_{\bullet}\left(\mathbb{RP}^2;\mathbb{Z}_2\right) \\ =& \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \end{aligned}$$

so

$$\dim J^{\#}(\circ) = 3 = \mathsf{Tait}(\circ)$$

Thank you!



https://community.wolfram.com/groups/-/m/t/1078687