Vector Geometry

This material corresponds roughly to sections 12.1, 12.2, 12.3 and 12.4 in the book.

Vectors:

- These are new mathematical objects which we will typically denote in bold letters, like $\mathbf{v}$, or with an arrow above them, like $\overrightarrow{v}$. Graphically they are represented by arrows.
- Vectors are made of two pieces of data: a) its magnitude, also called the **norm** of the vector, which is a non-negative number typically denoted, $|\mathbf{v}|$, $\|\mathbf{v}\|$ or $\|\overrightarrow{v}\|$, and b) a sense of direction. The latter can be specified in different ways but at least when the vector is drawn on the xy plane the direction is given by the angle between the vector $\mathbf{v}$ and the x axis.
- Vectors can be drawn anywhere on the plane and they are regarded the same when they have the same size (i.e, norm) and point in the same direction.

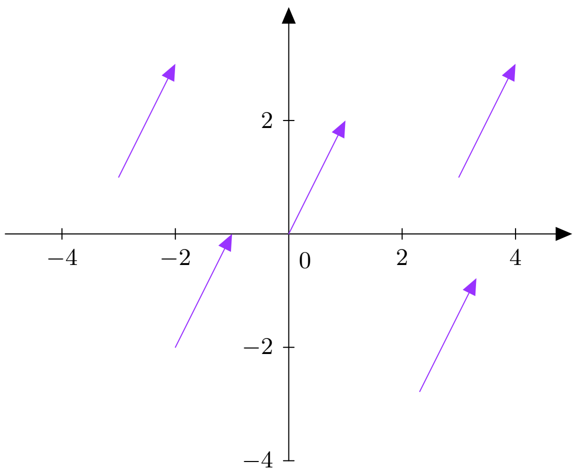

Figura 1: All of these vectors are considered the same.

Scalars and multiplication with vectors

- In this course the new mathematical jargon is to call a real number a **scalar.** Typically the letter λ will denote a scalar (i.e, a real number).
- Given a scalar λ and a vector $\mathbf{v}$, it is possible to multiply them and obtain a *new* vector, which is denoted $\lambda\mathbf{v}$.
- For example, if $\lambda = 3$, then $3\mathbf{v}$ represents a vector which is three times as long as $\mathbf{v}$, in other words $\|3\mathbf{v}\| = 3\|\mathbf{v}\|$, and which points in the same direction as $\mathbf{v}$.
- On the other hand, if $\lambda = -2$, then $-2\mathbf{v}$ represents a vector which is twice as long as $\mathbf{v}$, so $\|-2\mathbf{v}\| = 2\|\mathbf{v}\|$, but which points in the *opposite* direction as $\mathbf{v}$.
- When $\lambda = 0$, then $0\mathbf{v}$ gives you the zero or null vector $\mathbf{0}$, that is, the only vector with no length, which is drawn as a point. The null vector $\mathbf{0}$ is the only vector for which we do not try to assign a specific direction.
- You may have already figured out that the norm of a vector $\|\mathbf{v}\|$ behaves in many ways like an absolute value. For example,

$$\|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\| \tag{1}$$

is the analogue of the property $|ab| = |a||b|$. As a consequence, whenever $-1 < \lambda < 1$, the vector $\lambda\mathbf{v}$ will be smaller than $\mathbf{v}$, while if $|\lambda| \geq 1$, the vector $\lambda\mathbf{v}$ will be larger (or equal in size) to $\mathbf{v}$.

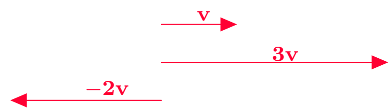

Figura 2: Multiplying a scalar with a vector

Addition and subtraction of vectors

- In the same way in which we can add numbers, it is possible to add two vectors $\mathbf{a}$ and $\mathbf{b}$ and produce a new vector $\mathbf{a}+\mathbf{b}$.

- To find $\mathbf{a}+\mathbf{b}$, you move $\mathbf{a}$ and $\mathbf{b}$ so that the head of $\mathbf{a}$ coincides with the tail of $\mathbf{b}$, and then $\mathbf{a}+\mathbf{b}$ will correspond to the arrow that starts at the tail of $\mathbf{a}$ and ends at the head of $\mathbf{b}$, as shown in the figure. Although it is not immediately obvious, notice that doing $\mathbf{b}+\mathbf{a}$ actually produces the same vector, in other words

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} \tag{2}$$

- Geometrically we can think of $\mathbf{a}$ and $\mathbf{b}$ as representing the two sides of a parallelogram, in which case $\mathbf{a}+\mathbf{b}$ corresponds to one of the diagonals.

- To find $\mathbf{a}-\mathbf{b}$ notice that this is the same as adding $\mathbf{a}$ with $-\mathbf{b}$, in other words, doing $\mathbf{a}+(-\mathbf{b})$. In fact, this corresponds to the other diagonal of the parallelogram.

- Moreover, notice that in general $\|\mathbf{a}\pm\mathbf{b}\|$ will be different from $\|\mathbf{a}\|\pm\|\mathbf{b}\|$!

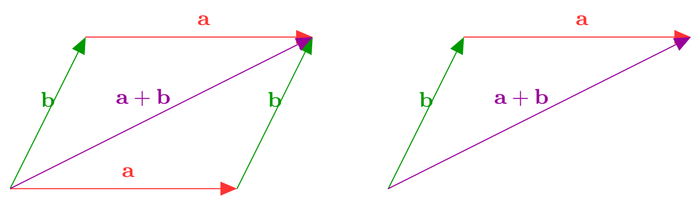

Figura 3: Adding Vectors

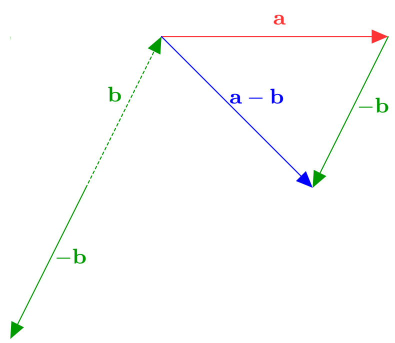

Figura 4: Subtracting vectors

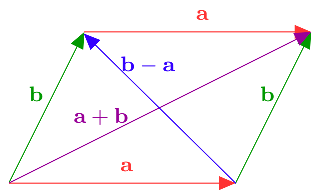

Figura 5: Addition and subtraction of vectors corresponds to the diagonals of the parallelogram

Vectors and Coordinate Systems

- A powerful way to study vectors is by using a coordinate system. Namely, one draws the xy plane and identifies the null vector $\mathbf{0}$ with the origin of your coordinate system.

- We can then represent any vector $\mathbf{v}$ in such a way that the tail of the vector $\mathbf{v}$ coincides with $\mathbf{0}$ and its head has coordinates (x, y). In this way the vector $\mathbf{v}$ can be thought of as a **position vector,** that is, a vector which specifies the position of a point (x, y). When we think of a vector as representing the position vector of some point Q, we will typically write this vector as $\mathbf{r}_Q$ instead of our usual notation $\mathbf{v}$. From this perspective, notice that the subtraction of two position vectors $\mathbf{r}_Q - \mathbf{r}_P$ corresponds to the net displacement vector, that is, a vector that starts at point P and ends at point Q. We will tend to use the notation

$$\overrightarrow{PQ} = \mathbf{r}_{PQ} = Q - P = \mathbf{r}_Q - \mathbf{r}_P \quad (3)$$

  for such a displacement vector.

- More generally, we can do this for vectors in three dimensional space, where we are now using three coordinate axes x, y, z. In this case the vector $\mathbf{v}$ is identified with the position vector which starts at the origin and ends at the point (x, y, z).

- We can also choose to stop drawing the entire arrow and just focus on the coordinates (x, y, z) of the head of the vector. Under this interpretation**, we can think of vectors as corresponding to points in space.**

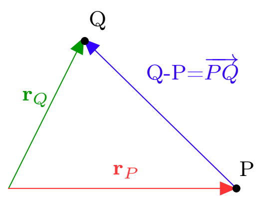

Figura 6: Subtraction of position vectors corresponds to a net displacement

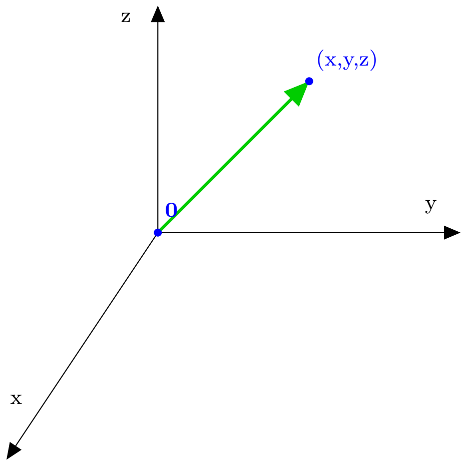

Figura 7: Vector **v** $\iff$ position vector starting from the origin

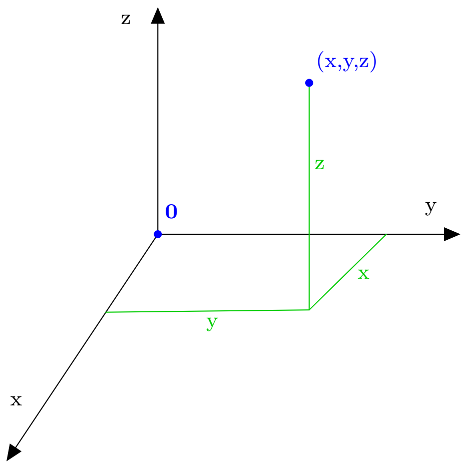

Figura 8: Vector $\iff$ point (x, y, z) in space

Interpreting an algebraic equation geometrically:
Depending on the number of variables one is considering, the solutions to an equation may change. For example, the solution to the equation $x = 0$ is:

- A point if x is the only variable [so the origin in the number line].
- A line if x, y are the variables [so the y axis in the xy plane]
- A plane if x, y, z are the variables [so the yz plane in the xyz 3d space]

Basis vectors for a cartesian coordinate system

- Every cartesian coordinate systems comes equipped with a standard set of vectors, which are known as the basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.
- Each of these basis vectors has unit length, that is,

$$\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1 \tag{4}$$

and they point in the direction of the x, y and z axes respectively. When we identify a vector with a point we can write

$$\mathbf{i} = (1, 0, 0)$$
$$\mathbf{j} = (0, 1, 0)$$
$$\mathbf{k} = (0, 0, 1)$$

- The key feature of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is that if the vector $\mathbf{v}$ represents the point (x, y, z), then it can be decomposed in an "obvious" way in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Namely, we have

$$\mathbf{v} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{5}$$

- For example, the vector $(1, 0, -\pi)$ decomposes as

$$(1, 0, -\pi) = \mathbf{i} + 0\mathbf{j} - \pi\mathbf{k} = \mathbf{i} - \pi\mathbf{k} \tag{6}$$

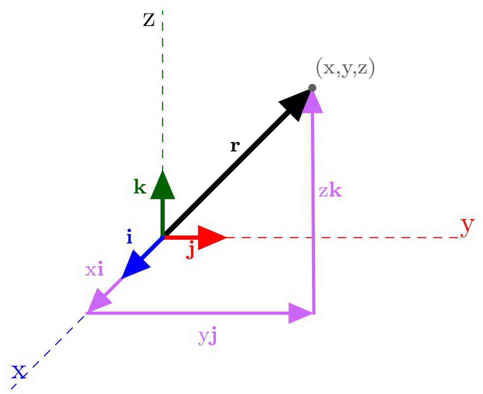

Figura 9: Basis vectors for cartesian coordinates

Dot product

- The dot product is an operation that takes two vectors $\mathbf{a}$ and $\mathbf{b}$, and produces a *scalar,* which we denote $\mathbf{a}\cdot\mathbf{b}$. Geometrically it is computed as

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \tag{7}$$

where θ represents the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. In general there are two such angles between the vectors, but we use the smallest one, that is, the angle θ between 0 and 180 degrees. In particular,

$$\mathbf{a}\cdot\mathbf{b} = 0 \iff \mathbf{a},\mathbf{b} \text{ are perpendicular} \tag{8}$$

A more standard name for perpendicular vectors in this course will be **orthogonal vectors.** So, two vectors are orthogonal with respect to one another if their dot product is zero. It is useful to notice then that for the basis vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ we have

$$\begin{aligned} \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \\ \mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0 \end{aligned}$$

- More generally, $\mathbf{a}\cdot\mathbf{b} > 0$ if and only if the angle between $\mathbf{a}$ and $\mathbf{b}$ is acute, while $\mathbf{a}\cdot\mathbf{b} < 0$ if and only if the angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.

- From the geometric definition of the dot product we can also see that

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} \tag{9}$$

- It is also possible to compute the dot product using a cartesian coordinate system. Namely, if we write $\mathbf{a}$ and $\mathbf{b}$ as

$$\begin{aligned} \mathbf{a} &= (a_1, a_2, a_3) \\ \mathbf{b} &= (b_1, b_2, b_3) \end{aligned}$$

then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 \tag{10}$$

so the dot product is the result of adding the individual products between the corresponding entries.

- For example, if $\mathbf{a} = (2, -1, e)$ and $\mathbf{b} = (1, 7, 0)$ then

$$\mathbf{a}\cdot\mathbf{b} = 2\cdot 1 + (-1)\cdot 7 + e\cdot 0 = 2 - 7 = -5 \tag{11}$$

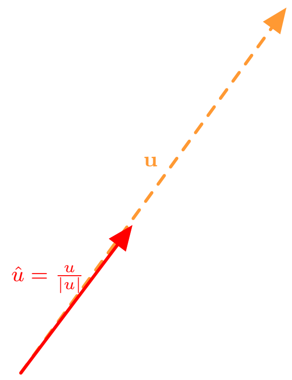

Figura 10: Unitary vector determined by $\mathbf{u}$

Unitary Vectors

- Each non-zero vector $\mathbf{u}$ determines a **unitary vector** $\hat{\mathbf{u}}$: this is a vector of unit length that points in the same direction as $\mathbf{u}$. It is not hard to see that $\hat{\mathbf{u}}$ is basically a rescaled version of $\mathbf{u}$, namely

$$\hat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} \tag{12}$$

The last equation can also be re-written as

$$\mathbf{u} = \|\mathbf{u}\|\hat{\mathbf{u}} \tag{13}$$

- One reason unitary vectors are useful is because of dimensional analysis. Namely, sometimes we want to think of a vector as carrying units. For example, $\mathbf{u}$ could have units of length/time if we think of it as a velocity vector. In this case, $\hat{\mathbf{u}}$ will have no units, that is, it is dimensionless.

Cross product

- The cross product is an operation that takes two vectors $\mathbf{a}$ and $\mathbf{b}$, and produces a *vector*, which we denote $\mathbf{a} \times \mathbf{b}$. Geometrically $\mathbf{a} \times \mathbf{b}$ will be a vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ and of magnitude

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta \tag{14}$$

where θ represents the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Notice that there are two natural choices for what the direction of $\mathbf{a} \times \mathbf{b}$ should be. The standard convention is to use the **right hand rule** to choose the direction of $\mathbf{a} \times \mathbf{b}$. Observe also that

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \iff \mathbf{a}, \mathbf{b} \text{ are parallel (or anti-parallel)} \tag{15}$$

and that $\|\mathbf{a} \times \mathbf{b}\|$ gives the area of the parallelogram whose sides correspond to $\mathbf{a}$ and $\mathbf{b}$.

- It is useful to notice then that for the basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ we have

$$\begin{aligned} \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} &= \mathbf{0} \\ \mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} \end{aligned}$$

The last three formulas can be remembered using the cylic rule indicated in the figures.

- From the right hand rule we can also see that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} \tag{16}$$

In other words, reversing the order of the vectors in the cross product flips the sign!

- It is also possible to compute the cross product using a cartesian coordinate system. Namely, if we write $\mathbf{a}$ and $\mathbf{b}$ as

$$\begin{aligned} \mathbf{a} &= (a_1, a_2, a_3) \\ \mathbf{b} &= (b_1, b_2, b_3) \end{aligned}$$

then

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\,\mathbf{i} + (a_3 b_1 - a_1 b_3)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k} \tag{17}$$

- For example, if $\mathbf{a} = (2, -1, e)$ and $\mathbf{b} = (1, 7, 0)$ then

$$\mathbf{a} \times \mathbf{b} = ((-1)\cdot 0 - e\cdot(7))\mathbf{i} + (e\cdot 1 - 2\cdot 0)\mathbf{j} + (2\cdot 7 - (-1)\cdot 1)\mathbf{k} = -7e\mathbf{i} + e\mathbf{j} + 15\mathbf{k} \tag{18}$$

- A way to remember the formula for the cross product is using the "determinant" formula

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{19}$$

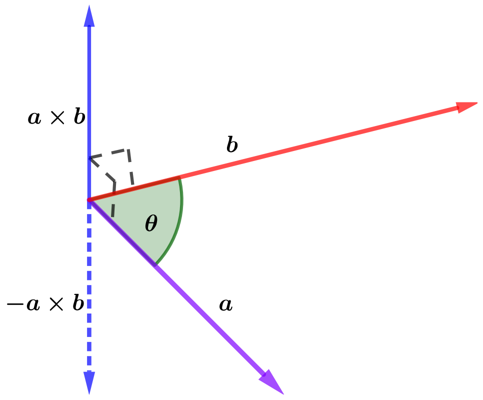

Figure 11: Cross product

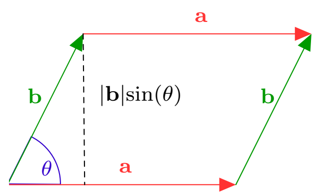

Figura 12: $\|\mathbf{a} \times \mathbf{b}\|$ is the area of the parallelogram with sides $\mathbf{a}, \mathbf{b}$

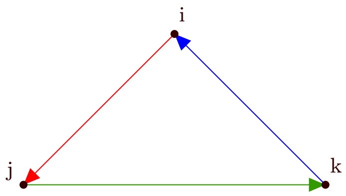

Figura 13: Cyclic property cross product

Dot product and Projecting Vectors

- Another application of the dot product is to find the projection (or shadow) of a vector $\mathbf{a}$ with respect to a vector $\mathbf{b}$.
- As shown in the next images, the idea is that we can two break or decompose $\mathbf{a}$ as the sum of two vectors $\mathbf{a}_{||}$ and $\mathbf{a}_{\perp}$

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_{\perp} \tag{20}$$

where: i) $\mathbf{a}_{||}$ is parallel to the vector $\mathbf{b}$, ii) $\mathbf{a}_{\perp}$ is orthogonal to the vector $\mathbf{b}$.

- The formula for the dot product allows us to write

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \tag{21}$$

so this equation together with the definition of the unitary vector $\hat{\mathbf{b}}$ tells us that

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{a}_{||} = (\|\mathbf{a}\|\cos\theta)\,\hat{\mathbf{b}} = \left(\|\mathbf{a}\|\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)\left(\frac{1}{\|\mathbf{b}\|}\mathbf{b}\right) = \frac{(\mathbf{a}\cdot\mathbf{b})}{\|\mathbf{b}\|^2}\mathbf{b} \tag{22}$$

- There is no separate formula for $\mathbf{a}_{\perp}$. Rather, one simply computes this vector as $\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{||}$. In other words

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{a}_{||} = \frac{(\mathbf{a}\cdot\mathbf{b})}{\|\mathbf{b}\|^2}\mathbf{b}$$
$$\mathbf{a}_{\perp} = \mathbf{a} - \frac{(\mathbf{a}\cdot\mathbf{b})}{\|\mathbf{b}\|^2}\mathbf{b}$$

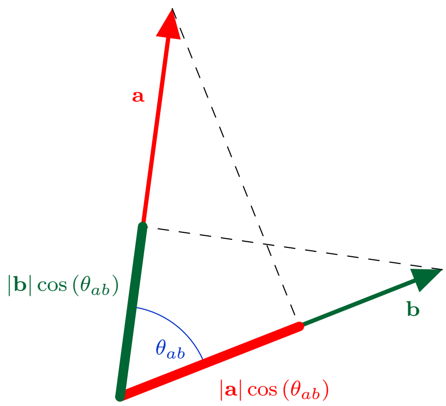

Figure 14: Dot Product and Projection of Vectors

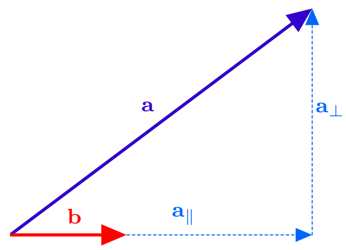

Figura 15: Projecting **a** along **b**

Distance between two points

- A final application of the dot product is to compute distances between points.
- Namely, recall that if P and Q are two points, then we have corresponding position vectors $\mathbf{r}_P$ and $\mathbf{r}_Q$. Moreover,

$$\overrightarrow{PQ} = Q - P = \mathbf{r}_Q - \mathbf{r}_P \tag{23}$$

  computes the displacement vector. Therefore, the norm of this vector will give us the distance between P and Q.
- Observe that from the formula for the dot product we have that

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|\|\mathbf{v}\| \cos\theta = \|\mathbf{v}\|^2 \cos 0 = \|\mathbf{v}\|^2 \tag{24}$$

  since the angle of any vector with itself is always 0. In particular, this means that for any vector $\mathbf{v}$ we can compute its magnitude as

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \tag{25}$$

- As a special case of this formula we have

$$\text{dist}(P, Q) = \|\overrightarrow{PQ}\| = \sqrt{\overrightarrow{PQ} \cdot \overrightarrow{PQ}} \tag{26}$$