MATH 251, Mariano Echeverria

Scalar Fields

This material corresponds roughly to sections 14.1, 14.2, 14.3, 14.4, 14.5 and 14.6 in the book.

Finding the graph of a function $z(x, y)$:

1. Draw the cartesian coordinate system xyz: its points are triples (x, y, z)
2. For each point $(x, y, 0)$, that is, a point (x, y) on the xy plane at height 0, find the value $z(x, y)$ and draw the point $(x, y, z(x, y))$
3. The graph of $z(x, y)$ is the surface obtained by applying the procedure in step 2 to each point in the domain of $z(x, y)$

Finding the domain of a function $z(x, y)$:
This works in a very similar way to how you find domains of a function of a single variable. For example, for the domain of

$$z(x, y) = \ln(1 - x - 2y) \tag{1}$$

You require that

$$1 - x - 2y > 0 \tag{2}$$

since ln is only defined for positive numbers. Similarly, for

$$z(x, y) = \frac{e^x}{x - y} \tag{3}$$

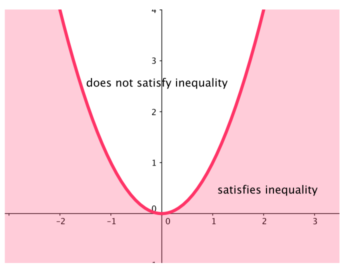
You require that $x \neq y$ since in general a fraction becomes undefined when the denominator is 0. The biggest difference with the case of functions of one variable is that the domain will have a more interesting geometric shape (for example, circles, planes, deleted lines) rather than intervals.

Solving an inequality $f(x,y) \geq 0$

1. Find the equation $f(x,y) = 0$. For example, if you want to determine when $x^2 - y \geq 0$ you find first the equation
$$x^2 = y \tag{4}$$
which is the equation of a parabola.

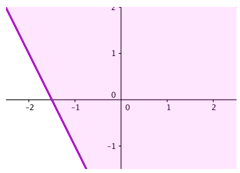
2. The equation divides the plane into different regions. On each region choose a point (a,b) and compute $f(a,b)$. If $f(a,b) \geq 0$ then that region satisfies the inequality, otherwise it does not satisfy it. For example, for $x^2 - y \geq 0$ we have $f(x,y) = x^2 - y$ and we choose the points $(0,1)$ and $(0,-1)$. Since $f(0,1) = -1$ which is not positive this region does not satisfy the inequality while $f(0,-1) = 1$ is positive so the second region does satisfy it.

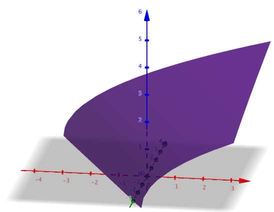
A similar strategy works for $f(x,y) < 0$.



Problem 1. Find the domain of the function $z(x,y) = \sqrt{2x + y + 3}$ and represent it on the xy plane.

Since we can only take square roots of nonnegative numbers we require
$$2x + y + 3 \geq 0 \tag{5}$$
To represent the points (x,y) that satisfy this inequality we represent first the equation
$$2x + y + 3 = 0 \tag{6}$$
on the xy plane. This is the equation of the straight line
$$y = -2x - 3 \tag{7}$$
and it separates the xy plane into two regions. All the points that satisfy the inequality correspond to only one of the regions. To determine which region satisfies the inequality choose a random point, for example $(0,0)$. It is easy to see that $(0,0)$ satisfies the inequality $2x + y + 3 \geq 0$ so the region that works must contain the origin as shown in the next figure (the graph is also shown although you won't need to do it).





Problem 2. Find the domain of $f(x,y) = \frac{\sqrt{4-x^2-y^2}}{x-y}$ and represent it on the xy plane.

In this case we need the expression inside the square root to be nonnegative, that is,

$$4 - x^2 - y^2 \geq 0 \tag{8}$$

Again to plot the inequality on the xy plane we fist plot the equality which is

$$4 - x^2 - y^2 = 0 \implies x^2 + y^2 = 4 \tag{9}$$

and this is the equation of a circle of radius 2 centered at the origin. Again, the circle divides the xy plane into two regions and all the points satisfying the inequality belong to one of the regions. It is easy to see that $(0,0)$ satisfies the inequality $4 - x^2 - y^2 \geq 0$, so the interior of the circle must correspond to the region which satisfies the inequality.

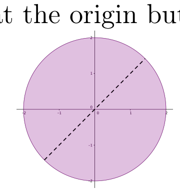
Also, we need the denominator to be different from zero, that is

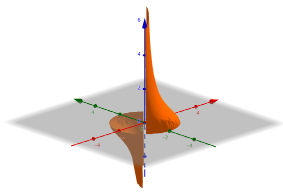
$$x - y \neq 0 \tag{10}$$

This means that the domain cannot include the equation

$$x = y \tag{11}$$

And so the domain must be all points (x,y) which belong to the disk of radius 2 centered at the origin but which do not lie on the straight line $x = y$.





Finding Level Curves: Suppose that $z(x,y)$ is a function of two variables x and y. If c is some value of the function z , then the equation

$$z(x,y) = c \tag{12}$$

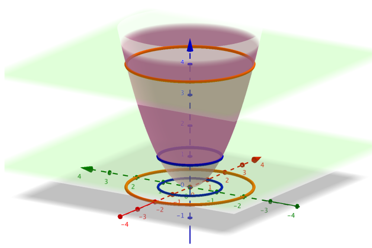
describes a curve lying on the plane $z = c$ called the **trace** of the graph of z in the plane $z = c$.
If this trace is projected onto the xy plane, the resulting curve in the xy plane is called a **level curve.** By drawing the level curves corresponding to several admissible values of c, we obtain a **contour map**.

Problem 3. Find the level curves of $z(x,y) = x^2 + y^2$

We need to solve the equation

$$x^2 + y^2 = c \tag{13}$$

Observe that if $c < 0$ the previous equation has no solution. If $c \geq 0$, the equation corresponds to a circle centered at $(0,0)$ of radius $\sqrt{c}$ and so the level curves are concentric circles.



Problem 4. Sketch the level curves of $f(x,y) = \frac{y}{x^2+1}$ corresponding to $z = -1, 0, 1$.

The intersection of $z = f(x,y) = \frac{y}{x^2+1}$ with the plane $z = -1$ are the points satisfying the equation

$$-1 = \frac{y}{x^2+1} \implies -x^2 - 1 = y \tag{14}$$

which is the equation of a parabola.

The intersection of $z = f(x,y) = \frac{y}{x^2+1}$ with the plane $z = 0$ are the points satisfying the equation

$$0 = \frac{y}{x^2+1} \implies 0 = y \tag{15}$$

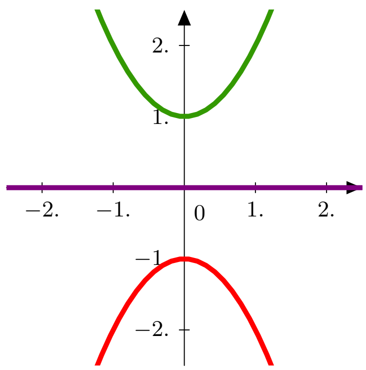
which is the equation of the x axis.

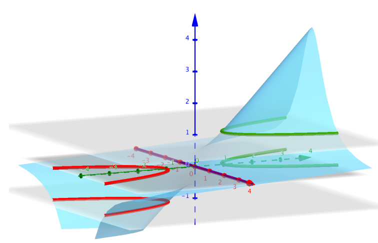
The intersection of $z = f(x, y) = \frac{y}{x^2+1}$ with the plane $z = 1$ are the points satisfying the equation

$$1 = \frac{y}{x^2 + 1} \implies x^2 + 1 = y \tag{16}$$

which is the equation of a parabola.

The following image shows the level curves of $f(x, y)$ corresponding to $z = -1, 0, 1$





Limits and continuity [this topic will not be evaluated on the exam, only on the written homework ☺]

We say that

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \tag{17}$$

If f is defined near (x_0, y_0), although not necessarily at the point (x_0, y_0), and for every $\epsilon > 0$ there exists $\delta > 0$ such that if (x, y) satisfies

$$0 < \|(x, y) - (x_0, y_0)\| < \delta \tag{18}$$

then

$$|f(x, y) - L| < \epsilon \tag{19}$$

We sat that f is **continuous** at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0) \tag{20}$$

Example 5. Analyze the limit

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^3 y}{x^6 + y^2} \tag{21}$$

Notice that the domain of this function is every point in the plane except for the origin, that is,

$$U = \left\{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\right\} \tag{22}$$

For the limit to exist it must do so for every direction which choose when approaching the point in question. In particular, we can use a line of slope m to approach the origin. Namely, consider the points of the form $(x,y) = (x, mx)$ and notice that on this points $f(x,y) = \frac{x^3 y}{x^6+y^2}$ behaves as

$$\frac{x^3 y}{x^6 + y^2} = \frac{m x^4}{x^6 + m^2 x^2} = \frac{m x^2}{x^4 + m^2} \longrightarrow 0 \tag{23}$$

as $x \longrightarrow 0$. On the other hand, if one takes the curve $y = x^3$ to approach the origin we find that

$$\frac{x^3 y}{x^6 + y^2} = \frac{x^6}{x^6 + x^6} = \frac{1}{2} \tag{24}$$

Since the limit must be independent of the curve chosen to approach $(0,0)$ we conclude that the limit does not exist.

Example 6. Determine if the function

$$f(x,y) = \begin{cases} \frac{y}{y+x^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \tag{25}$$

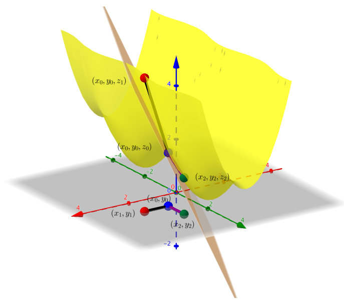
is continuous at the origin.

As in the previous example, we choose lines with slope m. If $y = mx$ then

$$\frac{y}{y + x^2} = \frac{mx}{mx + x^2} = \frac{m}{m + x} \longrightarrow 1 \tag{26}$$

Which means that even if the limit exists (which in fact it doesn't as one can verify by taking $y = x^2$ instead!) it would not agree with $f(0,0) = 0$, so we conclude that f is not continuous at the origin.

Partial Derivative of a function $z = f(x, y)$



- Recall that the graph of f can be represented by a surface. To specify the tangent plane to the surface passing through the point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$, we need two "slopes" which describe how the tangent plane is tilted with respect to the xy plane. These slopes are the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

- In practice, they can be computed by treating the other variables as constants. $\frac{\partial f}{\partial x}$ indicates how much the function f is changing when you move in a direction parallel to the x axis. $\frac{\partial f}{\partial y}$ indicate how much the function f is changing when you move in a direction parallel to the y axis. They are also denoted f_x and f_y respectively.

For example, if $f(x, y) = x^2 y + 3 \sin x$ and we want to find $\frac{\partial f}{\partial x}$ then we consider y as a constant and differentiate the function as if it is a function depending exclusively on x and so

$$\frac{\partial f}{\partial x} = (2x)y + 3 \cos x \tag{27}$$

Similarly, to find $\frac{\partial f}{\partial y}$ we consider x as a constant and differentiate the function as if it is a function depending exclusively on y and so

$$\frac{\partial f}{\partial y} = x^2 \tag{28}$$

Problem 7. Find $f_x(1, 2)$ and $f_y(1, 2)$ if $f(x, y) = e^{xy}$.
The partial derivatives of f are

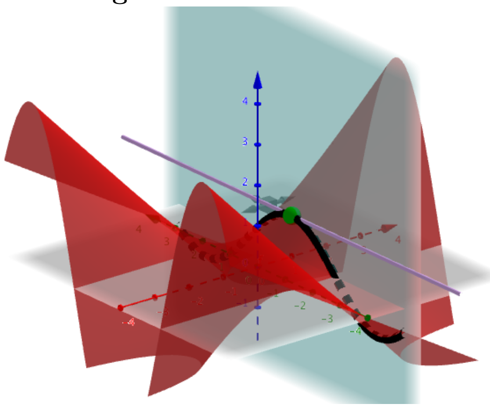
$$\begin{aligned} \frac{\partial f}{\partial x} &= y e^{xy} \\ \frac{\partial f}{\partial y} &= x e^{xy} \end{aligned} \tag{29}$$

Therefore

$$\begin{aligned} f_x(1, 2) &= 2e^2 \\ f_y(1, 2) &= e^2 \end{aligned} \tag{30}$$

Example 8. The intersection of the plane $x = 1$ with the graph of $f(x, y) = x \cos(y)$ gives you a curve as indicate in the following image. Find the slope

of the tangent line to this curve when the value of y is 0.



At the intersection between the plane and the graph of f we have the curve given by $f(1, y) = 1 \cdot \cos(y) = \cos y$. We want to find $f_y(1, 0)$ which is

$$f_y\,(1,0) = -\sin(y)\,|_{y=0} = 0 \tag{31}$$

Therefore, the slope of the tangent line is $m = 0$.

Gradient operator and directional derivatives

- It is also possible to analyze the rate of change of a function $T(x, y, z)$ in a direction which is not necessarily parallel to either the x, y, or z axis. That is, a rate of change which is not equal to $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$ or $\frac{\partial T}{\partial z}$. For example, we could try to analyze the rate of change of T when we move along the line through the origin determined by the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 1, 1)$. One could define a new notation and write something like $\frac{\partial T}{\partial \mathbf{v}}$. Rather than doing this, we define the **direction derivative of the scalar field T along the direction $\hat{\mathbf{v}}$** as the quantity

$$D_{\hat{\mathbf{v}}}T(x, y, z) = \lim_{\triangle t \to 0} \frac{T((x, y, z) + \hat{\mathbf{v}}\triangle t) - T(x, y, z)}{\triangle t} \tag{32}$$

  Notice that we used the normalized version of $\mathbf{v}$, that is, the unitary vector $\hat{\mathbf{v}}$ associated to $\mathbf{v}$.

- It is possible to determine this directional derivative in terms of the partial derivatives $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$ or $\frac{\partial T}{\partial z}$.

- First, given T we define a vector field called **the gradient of T,** which is denoted ∇T and it is defined as

$$\boldsymbol{\nabla} T(x, y, z) = \frac{\partial T(x, y, z)}{\partial x}\mathbf{i} + \frac{\partial T(x, y, z)}{\partial y}\mathbf{j} + \frac{\partial T(x, y, z)}{\partial z}\mathbf{k} \tag{33}$$

- It is not difficult to check that in this case

$$D_{\hat{\mathbf{v}}}T(x_0, y_0, z_0) = \boldsymbol{\nabla} T(x_0, y_0, z_0) \cdot \hat{\mathbf{v}} \tag{34}$$

Example 9. Suppose that $T(x,y,z) = xy + x^3 \sin z$. Find $\nabla T(x,y,z)$ and the direction derivative along the vector $\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$ at the point $(1,1,0)$.

First we compute the partial derivatives

$$\begin{aligned} \frac{\partial T(x,y,z)}{\partial x} &= y + 3x^2 \sin z \\ \frac{\partial T(x,y,z)}{\partial y} &= x \\ \frac{\partial T(x,y,z)}{\partial z} &= x^3 \cos z \end{aligned} \tag{35}$$

Therefore the gradient is

$$\boldsymbol{\nabla} T(x,y,z) = \left(y + 3x^2 \sin z\right)\mathbf{i} + x\mathbf{j} + x^3 \cos z \mathbf{k} \tag{36}$$

Recall that to compute the directional derivative we need to use a unitary vector. Since

$$\|\mathbf{u}\| = \frac{3}{\sqrt{5}} \tag{37}$$

we normalize it and work instead with

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \tag{38}$$

Therefore, the directional derivative at $(1,1,0)$ along the vector $\widehat{\mathbf{u}}$ can be computed as 34

$$D_{\hat{\mathbf{u}}} T(1,1,0) = \boldsymbol{\nabla} T(1,1,0) \cdot \hat{\mathbf{u}} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) = \frac{1}{3} \tag{39}$$

Interpretation of the gradient ∇T

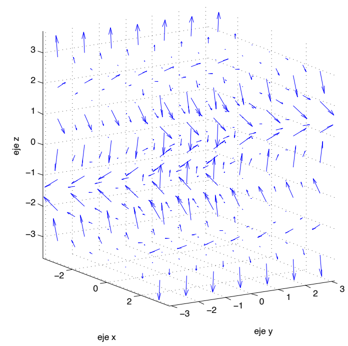
- Given that $D_{\hat{v}} T(x,y,z) = \nabla T(x,y,z) \cdot \hat{\mathbf{v}} = \|\nabla T(x,y,z)\| \cos(\theta)$, where θ is the angle between $\nabla T(x,y,z)$ and $\hat{\mathbf{v}}$, we can see (since $-1 \leq \cos\theta \leq 1$) that for any direction $\hat{\mathbf{v}}$, we have

$$-\|\nabla T(x,y,z)\| \leq D_{\hat{\mathbf{v}}} T(x,y,z) \leq \|\nabla T(x,y,z)\| \tag{40}$$

  From this we easily conclude that

- If $T(x,y,z)$ is a scalar field, then if one moves along the direction specified by $\nabla T(x,y,z)$, one experiences the largest increase of T. Likewise, moving in the direction determined by $-\nabla T(x,y,z)$, one experiences the largest decrease of T.

- **Moreover, if one moves in directions orthogonal to the gradient ∇T, then one does not experience any change of T.** That is, T is constant along those directions. For a function of three variables like $T(x,y,z)$, these regions where T does not change typically gives rise to surfaces known as **isothermal surfaces** (if one thinks of T as temperature), or **equipotential surfaces** (if one thinks of T as an electric potential).



Exercise 10. If $w = x^2 + xy + y^2 - z$**,** find at the point $(1, 1, 3)$ the highest rate of change of w and the direction along which that happens.

Here the scalar field is

$$w(x, y, z) = x^2 + xy + y^2 - z \tag{41}$$

We need to compute the gradient

$$\nabla w(x, y, z) = (2x + y)\,\mathbf{i} + (x + 2y)\,\mathbf{j} - \mathbf{k} \tag{42}$$

The direction where w increases at the highest rate is given by normalizing the gradient at the point $(1, 1, 3)$:

$$\hat{\mathbf{u}}(1, 1, 3) = \frac{\nabla w(1, 1, 3)}{|\nabla w(1, 1, 3)|} = \frac{3\mathbf{i} + 3\mathbf{j} - \mathbf{k}}{\sqrt{19}} \tag{43}$$

and the value of the rate of change at that point is:

$$D_{\hat{\mathbf{u}}} w\,(1, 1, 3) = |\nabla w(1, 1, 3)| = \sqrt{19} \tag{44}$$

Rules for gradients:

If $f(x,y,z)$ and $g(x,y,z)$ are two differentiable scalar field

- The sum $f + cg$ is again another scalar field, where c is a constant. Moreover,

$$\begin{gathered} \nabla\left(f + cg\right) = \nabla f + c\nabla g \\ D_{\hat{\mathbf{v}}}\left(f + cg\right)\left(\mathbf{r}\right) = D_{\hat{\mathbf{v}}} f\left(\mathbf{r}\right) + cD_{\hat{\mathbf{v}}} g\left(\mathbf{r}\right) \end{gathered} \tag{45}$$

- The product fg is a new scalar field and moreover

$$\begin{gathered} \nabla\left(fg\right) = \left(\nabla f\right)g + f\left(\nabla g\right) \\ D_{\hat{\mathbf{v}}}\left(fg\right)\left(\mathbf{r}\right) = \left(D_{\hat{\mathbf{v}}} f\left(\mathbf{r}\right)\right)g(\mathbf{r}) + f\left(\mathbf{r}\right)\left(\nabla_{\hat{\mathbf{v}}} g\left(\mathbf{r}\right)\right) \end{gathered} \tag{46}$$

- If the gradient of a scalar field is always null, that is

$$\nabla T(x,y,z) \equiv \mathbf{0} \quad \text{for all} \ \ (x,y,z) \tag{47}$$

  then T is a constant scalar field.

Higher Order Derivatives:

- We use the notation f_{xx} or $\frac{\partial^2 f}{\partial x^2}$ to indicate that we differentiate twice with respect to x.

- We use the notation f_{yy} or $\frac{\partial^2 f}{\partial y^2}$ to indicate that we differentiate twice with respect to y.

- We use the notation f_{xy} or $\frac{\partial^2 f}{\partial y \partial x}$ to indicate that we differentiate first with respect to x and then with respect to y

- We use the notation f_{yx} or $\frac{\partial^2 f}{\partial x \partial y}$ to indicate that we differentiate first with respect to y and then with respect to x

- There is no need to remember the order of x and y in the last two partial derivatives since they will agree for the functions we compute, that is, we will have
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \tag{48}$$
The mathematical way to say this is that the **mixed partial derivatives commute.**

As an example, if
$$f(x, y) = x \sin(x^2 + y) \tag{49}$$
Then
$$\frac{\partial f}{\partial x} = f_x = \sin(x^2 + y) + 2x^2 \cos(x^2 + y) \tag{50}$$
We can now consider f_x as a new function of two variables and take its derivatives with respect to x or y. For example
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = 2x\cos(x^2 + y) + 4x\cos(x^2 + y) - 4x^3 \sin(x^2 + y) \tag{51}$$
We can now differentiate with respect to y twice
$$\begin{aligned} \tfrac{\partial f}{\partial y} &= f_y = x\cos(x^2 + y) \\ \tfrac{\partial^2 f}{\partial y^2} &= f_{yy} = -x\sin(x^2 + y) \end{aligned} \tag{52}$$
Now we differentiate f with respect to x first and then with respect to y:
$$\begin{aligned} \tfrac{\partial}{\partial y}\left(\tfrac{\partial f}{\partial x}\right) &= \tfrac{\partial}{\partial y}\left(\sin(x^2 + y) + 2x^2\cos(x^2 + y)\right) \\ &= \cos(x^2 + y) - 2x^2 \sin(x^2 + y \end{aligned} \tag{53}$$
Similarly, we can differentiate with respect to y first and then with respect to x:
$$\begin{aligned} \tfrac{\partial}{\partial x}\left(\tfrac{\partial f}{\partial y}\right) &= \tfrac{\partial}{\partial x}\left(x\cos(x^2 + y)\right) \\ &= \cos(x^2 + y) - 2x^2\sin(x^2 + y) \end{aligned} \tag{54}$$
This shows (at least in this case) that the mixed partial derivatives commute as state earlier, that is,
$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \tag{55}$$

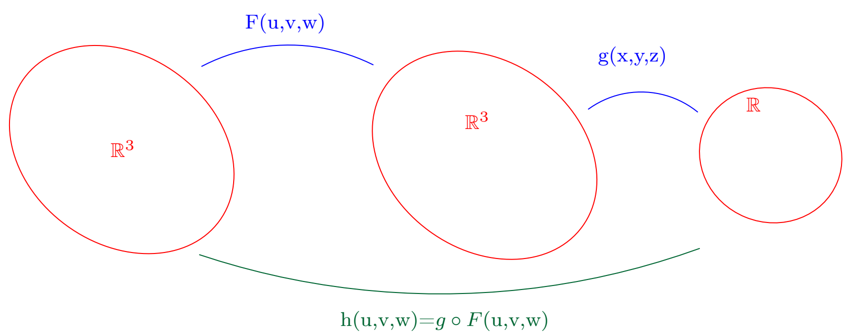


Figure 1: Composition of functions

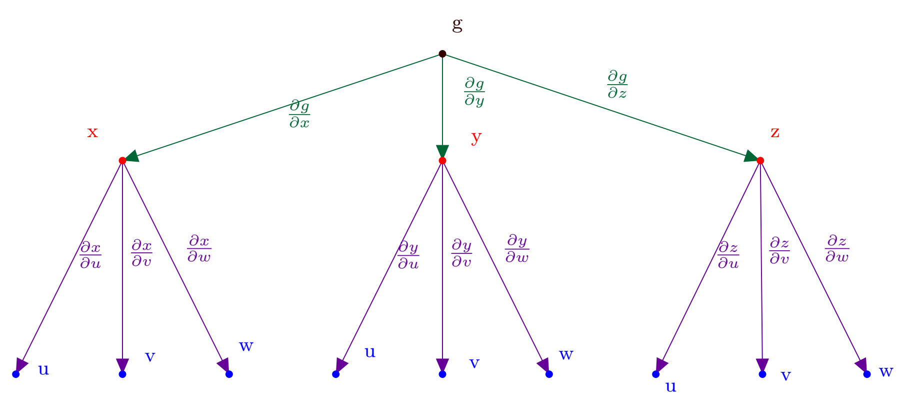


Figure 2: Tree diagram

Example 11. If $w = f(x,y)$ where $x = e^r \cos\theta$, $y = e^r \sin\theta$ verify the identity $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial \theta^2}\right)$

We use the following diagram for the chain rule

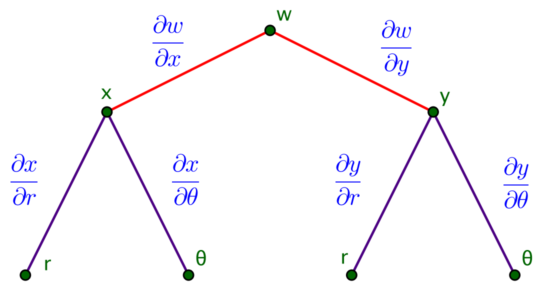


Figure 3: Tree diagram for chain rule

If we introduce also the notation

$$w_x \equiv \frac{\partial w}{\partial x} \quad w_y \equiv \frac{\partial w}{\partial y} \tag{56}$$

then the chain rule says that to compute something like w_r, we must take all paths from w to r and all the different contributions from each possible route. In other words,

$$w_r = w_x x_r + w_y y_r \tag{57}$$

Then w_{rr} is computed via the product rule

$$w_{rr} = \frac{\partial(w_x x_r + w_y y_r)}{\partial r} = w_{xr} x_r + w_x x_{rr} + w_{yr} y_r + w_y y_{rr} \tag{58}$$

To find $w_{xr} = \frac{\partial w_x}{\partial r}$ and $w_{yr} = \frac{\partial w_y}{\partial r}$ we can regard w_x, w_y as our new functions for the tree diagram so we put them on top (instead of w that is). Hence

$$\begin{aligned} w_{xr} &= w_{xx} x_r + w_{xy} y_r \\ w_{yr} &= w_{yx} x_r + w_{yy} y_r \end{aligned} \tag{59}$$

In this way

$$w_{rr} = \left(w_{xx} x_r + w_{xy} y_r\right) x_r + w_x x_{rr} + \left(w_{yx} x_r + w_{yy} y_r\right) y_r + w_y y_{rr} \tag{60}$$

Since the way in which r, θ appear is completely symmetric, to find $w_{\theta\theta}$ we can use the same expression if we substitute r for θ. In this way

$$\begin{aligned} w_{rr} &= \left(w_{xx} x_r + w_{xy} y_r\right) x_r + w_x x_{rr} + \left(w_{yx} x_r + w_{yy} y_r\right) y_r + w_y y_{rr} \\ w_{\theta\theta} &= \left(w_{xx} x_\theta + w_{xy} y_\theta\right) x_\theta + w_x x_{\theta\theta} + \left(w_{yx} x_\theta + w_{yy} y_\theta\right) y_\theta + w_y y_{\theta\theta} \end{aligned} \tag{61}$$

Finally, using that $x = e^r \cos\theta$, $y = e^r \sin\theta$ we find

$$\begin{aligned} w_{rr} &= \left(w_{xx} e^r \cos\theta + w_{xy} e^r \sin\theta\right) e^r \cos\theta + w_x e^r \cos\theta \\ &+ \left(w_{yx} e^r \cos\theta + w_{yy} e^r \sin\theta\right) e^r \sin\theta + w_y e^r \sin\theta \end{aligned} \tag{62}$$

$$\begin{aligned} w_{\theta\theta} &= \left(-w_{xx} e^r \sin\theta + w_{xy} e^r \cos\theta\right)\left(-e^r \sin\theta\right) - w_x e^r \cos\theta \\ &+ \left(-w_{yx} e^r \sin\theta + w_{yy} e^r \cos\theta\right)\left(e^r \cos\theta\right) - w_y e^r \sin\theta \end{aligned} \tag{63}$$

Adding both equations and recalling that $w_{xy} = w_{yx}$ we obtain

$$w_{rr} + w_{\theta\theta} = e^{2r}\left(w_{xx} + w_{yy}\right) \tag{64}$$

That is,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial \theta^2}\right) \tag{65}$$

which is what we were after.

Equation tangent plane

If $g(x, y, z) = 0$ represents the equation of a surface then ∇g is a vector orthogonal to the surface (recall the discussion from before where we said that when one moves in directions perpendicular to ∇g then g is constant).

Therefore, if $P = (x_0, y_0, z_0)$ is a point on the surface then the equation of the tangent plane to the surface at the point P has normal equation

$$(x, y, z) \cdot \nabla g\left(x_0, y_0, z_0\right) = \left(x_0, y_0, z_0\right) \cdot \nabla g\left(x_0, y_0, z_0\right) \tag{66}$$

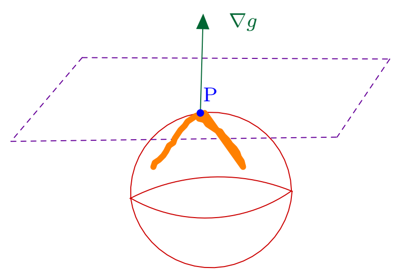


Figure 4: Tangent Plane

Example 12. The point $P = (2, 1, 5)$ belongs to the paraboloid $z = x^2 + y^2$. Find the equation of the tangent plane to the surface at that point.

We define

$$g(x, y, z) = z - x^2 - y^2 \tag{67}$$

and compute its gradient

$$\nabla g = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k} \tag{68}$$

Using equation 66 the normal form of the tangent plane is

$$(x, y, z) \cdot (-4, -2, 1) = (2, 1, 5) \cdot (-4, -2, 1) \tag{69}$$

which is the same as

$$-4x - 2y + z = -5 \tag{70}$$

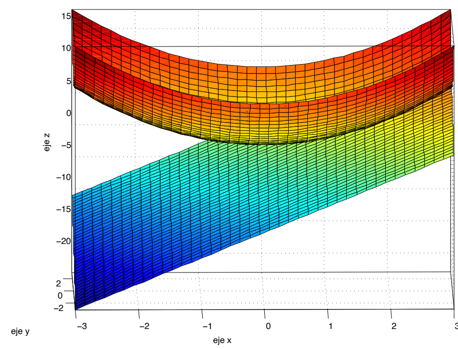


Figure 5: Tangent Plane

Example 13. The point $P = (0, 0, 0)$ belongs to the intersection of the paraboloid $z = x^2 + y^2 + x + 2y$ and the plane $z = 3x - 4y$**.** Find a vector tangent to the curve of intersection of both surfaces at the point P.

Let $g_1(x, y, z) = x^2 + y^2 + x + 2y - z = 0$ represent the first surface while $g_2(x, y, z) = 3x - 4y - z = 0$ the second one.

Call C the curve of intersection. Since C belongs to the first surface, its tangent vector must be orthogonal to $\nabla g_1(P)$. Likewise, its tangent vector must be orthogonal to $\nabla g_2(P)$. Given that

$$\nabla g_1 = (2x + 1)\,\mathbf{i} + (2y + 2)\,\mathbf{j} - \mathbf{k} \quad \nabla g_2 = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} \tag{71}$$

at the point $P = (0, 0, 0)$ the gradients are

$$\nabla g_1(P) = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \nabla g_2(P) = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} \tag{72}$$

And the vector $\mathbf{v}$ tangent to the curve will be orthogonal to both gradients, so we can take

$$\mathbf{v} = \nabla g_1(P) \times \nabla g_2(P) = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (3\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = -6\mathbf{i} - 2\mathbf{j} - 10\mathbf{k} \tag{73}$$

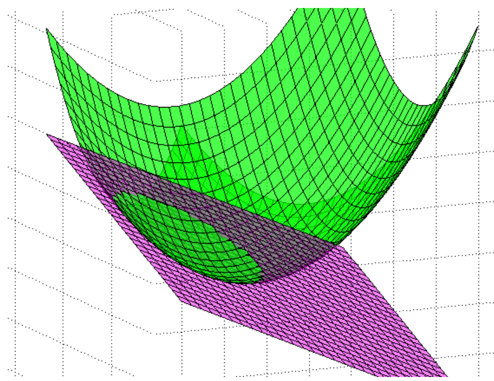


Figure 6: Intersection between plane and surface