Problems on Vector Geometry

This material corresponds roughly to sections 12.1, 12.2, 12.3 and 12.4 in the book, as well as the study guide Vector Geometry.

Test your understanding questions: the following questions are meant to summarize the key points from the sections covered in this study guide.

1. What is the line of intersection between the xz and yz planes?
2. What does the equation $x = 3$ correspond to if a) x is the only variable being considered, b) x, y are the only variables being considered, c) x, y, z are the only variables being considered.
3. What does the equation $x^2 + y^2 = 4$ correspond to if a) x, y are the only variables being considered, b) x, y, z are the only variables being considered.
4. Find the intersection between the sphere $3x^2+y^2+z^2 = 4$ and the cylinder $y^2+z^2 = 1$.
5. A vector $\mathbf{v}$ is called a unit vector if $|\mathbf{v}| = 1$. If $\mathbf{v}$ and $\mathbf{w}$ are unit vectors, is $\mathbf{v}+\mathbf{w}$ a unit vector? If not, what is the smallest and largest magnitude $|\mathbf{v}+\mathbf{w}|$ can take?
6. Suppose $\mathbf{v}$ is a vector on the xy plane. Then $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$ is known as the *polar decomposition* of the vector. Can you think of a reason why this name is given to the previous equation? Hint: think of complex numbers.
7. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$ are two vectors and $\mathbf{u}$ another vector such that $\text{proj}_{\mathbf{u}}\mathbf{v}_1 = 3\mathbf{i} - 5\mathbf{k}$ and $\text{proj}_{\mathbf{u}}\mathbf{v}_2 = 2\mathbf{j} + 7\mathbf{k}$. Find a) $\text{proj}_{\mathbf{u}}(\mathbf{v}_1 + 2\mathbf{v}_2)$, b) $\text{proj}_{\mathbf{u}}3\mathbf{v}_1$ and c) $\text{proj}_{3\mathbf{u}}\mathbf{v}_2$.
8. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are three different vectors. Does the expression $\mathbf{v}_1 \cdot \mathbf{v}_2 \cdot \mathbf{v}_3$ make sense? What about $\mathbf{v}_1 \times \mathbf{v}_2 \times \mathbf{v}_3$? What about $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$? Is the last expression the same as $\mathbf{v}_1 \times (\mathbf{v}_2 \cdot \mathbf{v}_3)$, or the same as $(\mathbf{v}_1 \cdot \mathbf{v}_2) \times \mathbf{v}_3$?
9. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then does $\mathbf{v} = \mathbf{w}$?

Problem 1. Find the vector v of norm 3 that makes an angle of $\frac{3\pi}{4}$ radians with the positive x axis.

The vector is

$$\mathbf{v} = 3\left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\right) = 3\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \tag{1}$$

Problem 2. Determine whether the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel, where $A = (0, 2, 5)$, $B = (3, 8, 2)$, $C = (3, 5, -7)$ and $D = (8, 15, -12)$

$$\begin{cases} \overrightarrow{AB} = B - A = (3, 6, -3) \\ \overrightarrow{CD} = D - C = (5, 10, -5) \end{cases} \tag{2}$$

If the vectors were parallel, one can find a real number such that

$$\overrightarrow{CD} = t\overrightarrow{AB} \tag{3}$$

In other words, we need to solve

$$\begin{cases} 5 = 3t \\ 10 = 6t \\ -5 = -3t \end{cases} \tag{4}$$

The solution of this system of equations is $t = \frac{5}{3}$ so the vectors are parallel. Alternatively, one can observe that

$$\overrightarrow{CD} \times \overrightarrow{AB} = (0, 0, 0) \tag{5}$$

which as I mentioned in class is another way of verifying whether they are parallel or not.

Problem 3. Consider $P = (1, 3)$, $Q = (5, -1)$, $R = (2, 3)$, $S = (x, 2)$ where x is unknown.

1) Find $\overrightarrow{PQ}$ and $\overrightarrow{RS}$

2) Find the value of x so that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ become parallel.

3) Find a unit vector in the direction of $\overrightarrow{RS}$

1)

$$\begin{cases} \overrightarrow{PQ} = Q - P = (4, -4) \\ \overrightarrow{RS} = S - R = (x - 2, -1) \end{cases} \tag{6}$$

2) we want to find a number t so that

$$(4, -4) = t(x - 2, -1) \tag{7}$$

This gives the equations

$$\begin{cases} 4 = t(x - 2) \\ -4 = -t \end{cases} \tag{8}$$

which has solution

$$x = 3 \tag{9}$$

3) the vector is

$$\hat{\mathbf{v}} = \frac{\overrightarrow{RS}}{\|\overrightarrow{RS}\|} = \frac{(1,-1)}{\sqrt{2}} \tag{10}$$

Problem 4. 1) Write the equation of a sphere with center $P = (1, -1, 3)$ and radius 2.

$$(x-1)^2 + (y+1)^2 + (z-3)^2 = 4 \tag{11}$$

2) Find the value (or values) of c such that $Q = (2, -1, c)$ is on the sphere. We need Q to satisfy the previous equation, that is,

$$(2-1)^2 + (-1+1)^2 + (c-3)^2 = 4 \tag{12}$$

which gives

$$c = 3 \pm \sqrt{3} \tag{13}$$

Problem 5. Take $P = (1, 1)$ and $\overrightarrow{PQ} = \langle -2, 3 \rangle$.

1. **Find point Q**
2. **What is the length of $\overrightarrow{PQ}$?**
3. **Find a unit vector parallel to $\overrightarrow{PQ}$.**

1) We need

$$Q - P = \overrightarrow{PQ} = (-2, 3) \tag{14}$$

from which we see that

$$Q = P + (-2, 3) = (-1, 4) \tag{15}$$

2) $\|\overrightarrow{PQ}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$

3) $\hat{\mathbf{v}} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{(-2,3)}{\sqrt{13}}$

Problem 6. Suppose that $\|\mathbf{v}\| = \sqrt{2}$ and $\mathbf{u}$ is a vector such that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{4}$ radians.

1) What must the value of $\|\mathbf{u}\|$ so that $\|\mathbf{u} + \mathbf{v}\| = \sqrt{5}$

2) Find the length of the orthogonal projection of $\mathbf{u}$ onto the line going through $\mathbf{v}$

1) Square both sides of the equation to obtain

$$\|\mathbf{u} + \mathbf{v}\|^2 = 5 \tag{16}$$

Now, notice that for any vector $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$ so taking $\mathbf{a} = \mathbf{u} + \mathbf{v}$ the left hand side can be rewritten as

$$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\frac{\pi}{4} + \|\mathbf{v}\|^2 \tag{17}$$

By assumption $\|\mathbf{v}\| = \sqrt{2}$ so we must solve the equation

$$\|\mathbf{u}\|^2 + 2\|\mathbf{u}\| + 2 = 5 \tag{18}$$

or

$$\|\mathbf{u}\|^2 + 2\|\mathbf{u}\| - 3 = 0 \tag{19}$$

whose only positive solution is

$$\|\mathbf{u}\| = 1 \tag{20}$$

2) Using the formula in the PDF (page 11) we need to find

$$\mathbf{u}_{\|} = \frac{(\mathbf{u}\cdot\mathbf{v})}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\|\mathbf{u}\|\|\mathbf{v}\|\cos\pi/4}{2}\mathbf{v} = \frac{1}{2}\mathbf{v} \tag{21}$$

so

$$\|\mathbf{u}_{\|}\| = \frac{1}{2}\|\mathbf{v}\| = \frac{\sqrt{2}}{2} \tag{22}$$

Problem 7. Consider the points $P = (1, 2, 0)$, $Q = (-1, 1, 1)$ and $R = (2, 0, 1)$.

a) Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$\overrightarrow{PQ} = Q - P = (-2, -1, 1)$ and $\overrightarrow{PR} = R - P = (1, -2, 1)$

b) Find the cross product $\overrightarrow{PQ} \times \left(\overrightarrow{PR} + 3\overrightarrow{PQ}\right)$

$$\overrightarrow{PQ} \times \left(\overrightarrow{PR} + 3\overrightarrow{PQ}\right) = \overrightarrow{PQ} \times \overrightarrow{PR} + 3\overrightarrow{PQ} \times \overrightarrow{PQ} = \overrightarrow{PQ} \times \overrightarrow{PR} = (1, 3, 5) \tag{23}$$

c) Find the area of the parallelogram whose sides are the vectors $2\overrightarrow{PQ}$ and $-3\overrightarrow{PR}$

The area is simply

$$\|2\overrightarrow{PQ} \times (-3\overrightarrow{PR})\| = 6\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = 6\|(1, 3, 5)\| = 6 \cdot \sqrt{35} \tag{24}$$

Problem 8. a) Consider the points $P = (1, 2, 1)$ and $Q = (x, 4, -x)$. Find the value (or values) of x such that $\left|\overrightarrow{PQ}\right| = \sqrt{14}$.

Squaring both sides we need to solve $\left|\overrightarrow{PQ}\right|^2 = 14$ which is the same as

$$(x-1)^2 + (4-2)^2 + (-x-1)^2 = 14 \tag{25}$$

and this gives the solution $x = \pm 2$.

b) Using the value (or values) found in a), find the vector projection of $\overrightarrow{PQ}$ along the vector $\mathbf{v} = (3, 0, 4)$, that is, find $\text{proj}_{\mathbf{v}}\overrightarrow{PQ}$

We need to compute

$$\frac{\overrightarrow{PQ}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{(x-1)3 + 0(2) + (-x-1)4}{25}(3, 0, 4) = \frac{-x-7}{25}(3, 0, 4) \tag{26}$$

depending on the value of x, we obtain $\frac{-9}{25}(3, 0, 4)$ or $\frac{-5}{25}(3, 0, 4)$.