Problems on Scalar Fields

This material corresponds roughly to sections 14.1, 14.2, 14.3, 14.4, 14.5 and 14.6 in the book.

Problem 1. **[Maxwell relation]** Thermodynamics teaches that the energy E of a rigid container of gas is a function of its entropy S and volume V: $E = E(S, V)$. Its temperature is given by $T = \frac{\partial E}{\partial S}$ and its pressure by $P = -\frac{\partial E}{\partial V}$. Show that $\frac{\partial T}{\partial V} = -\frac{\partial P}{\partial S}$. This is a Maxwell relation.

The relation is usually written $\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$ in physics and chemistry texts to make it clear that S is held constant on the left and V is held constant on the right when computing the partial derivatives.

Problem 2. **[Wave equation]** Let $u(x,t) = f(x - vt) + g(x + vt)$ where f, g are scalar valued functions (so they take real values). Show that

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{1}$$

by applying the chain rule (or tree diagrams). This is a *partial differential equation,* called the *wave equation.*

Problem 3. The equation $z = xf(y/x)$ defines a surface whenever $x \neq 0$ and f is a real valued function. Find the equation of the tangent plane to the surface passing through the point $(x_0, y_0, x_0 f(y_0/x_0))$. Does the origin $(0, 0, 0)$ belong to this plane?

Problem 4. Let $f(x,y) = x - y^2$ and $g(x,y) = 2x + \ln y$. Show that the level curves of f and g are orthogonal at every point where they meet.

Problem 5. If three resistors R_1, R_2, R_3 are connected in parallel, the total electrical resistance is determined by the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \tag{2}$$

Find $\frac{\partial R}{\partial R_1}$.

Problem 6. Suppose that a duck is swimming in a circle, $x = \cos t$, $y = \sin t$, while the water temperature is given by the formula $T = x^2 e^y - xy^3$. Find $\frac{dT}{dt}$ using the chain rule.

Problem 7. Show that the tangent plane at each point (x_0, y_0, z_0) of the cone $z = \sqrt{x^2 + y^2}$, (with $x_0 \neq 0$, $y_0 \neq 0$) contains the line passing through (x_0, y_0, z_0) and the origin.

Problem 8. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$.

a) Show that $\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$ whenever $r \neq 0$.

b) What is $\|\nabla\left(\frac{1}{r}\right)\|$?

c) In electrostatics, the force $\mathbf{F}_e$ of attraction between two particles of opposite charge is given by $\mathbf{F}_e = k\frac{\mathbf{r}}{r^3}$. A potential function V for the electrostatic force is a scalar function $V = V(x, y, z)$ such that $\nabla V = -\mathbf{F}_e$ (here I am using the physicist convention for the potential). Find a potential V for $\mathbf{F}_e$.

Problem 9. Show that the surface $x^2 - 2yz + y^3 = 4$ is perpendicular to any member of the family of surfaces $x^2 + 1 = (2 - 4a)y^2 + az^2$ at the point of intersection $(1, -1, 2)$.

Problem 10. Find the directional derivative of $U(x, y, z) = 2x^3y - 3y^2z$ at the point $P = (1, 2, -1)$ in a direction toward the point $Q = (3, -1, 5)$.