Problems Integration of Scalar Fields

This material corresponds roughly to sections 15.1, 15.2, 15.3, 12.7, 15.4, 15.6 and 16.4 in the book.

Problem 1. Consider

$$I = \int\int_R \cos\sqrt{y-x}\,dA \tag{1}$$

where R is the region determined by the curves $y = x+1$, $y = x^2 + x$. Find I using the change of variables $u = x$, $v = \sqrt{y-x}$

Problem 2. Find the volume of the solid of revolution given by the equation $z^2 \geq x^2 + y^2$, which is contained inside the sphere $x^2 + y^2 + z^2 = 1$

Problem 3. Prove Newton's Shell theorem for the gravitational potential. Namely, the gravitational potential created by an object with constant density ρ_M and spherically shaped on a point $(0,0,z_0)$ is

$$\int\int\int \frac{-Gdm}{\sqrt{x^2+y^2+(z-z_0)^2}} \tag{2}$$

where, $dm = \rho_M dvol$, G is Newton's universal gravitational constant, and the region of integration is the interior of the sphere of radius R centered at the origin. Use spherical coordinates to show that this integral equals

$$\begin{cases} -\frac{2}{3}\pi G\rho_M\left(3R^2 - z_0^2\right) & \textbf{if } 0 < z_0 \leq r \\ -\frac{GM}{z_0} & \textbf{if } r < z_0 \end{cases} \tag{3}$$

Problem 4. Consider the Gaussian integral

$$I_a = \int\int_D e^{-\left(x^2+y^2\right)}dxdy \tag{4}$$

where D is the disk $x^2 + y^2 \leq a^2$.

a) Use polar coordinates to show that $I_a = \pi\left(1 - e^{-a^2}\right)$.

b) Find $\int_0^{\infty} e^{-x^2} dx$ using the value of $\int\int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy$.

Example 5. Find the average value of the temperature $T(x, y, z) = x^2 + y^2 - z^2$ inside the interior of the region bounded by the surfaces $2z = x^2 + y^2$, $x^2 + y^2 - z^2 = 1$ and $z = 0$, $z = 3$. You can use that the average value of T, denoted $< T >$, is given by

$$< T > = \frac{\int\int\int_R T dV}{\mathbf{Vol}(R)} \tag{5}$$

Problem 6. Consider the region R determined by the surfaces $z = \sqrt{x^2 + y^2}$, $z = 2 - x^2 - y^2$. Write the integral for the volume of this region using cylindrical coordinates, first using the order of integration $dzdrd\theta$, and then using $drdzd\theta$. You do not need to compute the value of the integral!

Problem 7. Consider the double integral $I = \int_0^{\pi} \int_{\sin x}^{3+\cos(2x)} f(x, y) dydx$.

a) Draw the region of integration R.

b) Change the order of integration to $dxdy$. Do not compute the integral.

Problem 8. Consider the integral $I = \int\int\int_T \frac{xy+y^2}{x^3} dxdydz$, where T is the region inside the first octant $(x, y, z \geq 0)$ between the plane $x+y+z = 2$, the xy plane, and the vertical "walls" determined by the trapezoid given by the equations $x + y = 1$, $x + y = 2$, $y = 0$, $y = x$. As a suggestion, use the change of variables $x = \frac{v}{1+w}$, $y = \frac{vw}{1+w}$, $z = u - v$.

Problem 9. Consider the region determined by the surfaces $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 = 3z$.

a) Write an integral for the volume of this region using cylindrical coordinates, using the order $dzdrd\theta$.

b) Write the same integral now using spherical coordinates in the order $drd\varphi d\theta$, where φ represents the angle which starts from the z axis.

Problem 10. Make the change of variables $u = xy$, $v = \frac{y}{x}$ to find the volume of the solid bounded by the surfaces $z = x + y$, $xy = 1$, $xy = 2$, $y = x$, $y = 2x$, $z = 0$ $(x > 0,\ y > 0)$.

Problem 11. Consider the following triple integral

$$I = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} f(x,y,z)dzdydx \tag{6}$$

a) Draw the region of integration.
b) Write I in spherical coordinates, using the order of integration $drd\varphi d\theta$, and the order $d\varphi drd\theta$. In both cases θ is the angle that starts from the x axis.
