Integration of Vector Fields

This material corresponds roughly to sections 16.1, 16.2, 16.3, 17.1, 17.2, 17.3 in the book.

Line integral

If **F** is a vector field defined along a curve C , the **line integral** (or work) of the vector field along the curve is

$$\int_C \mathbf{F} \cdot d\mathbf{r} \tag{1}$$

Using a parametrization of the curve we can compute it as

$$\int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} dt \tag{2}$$

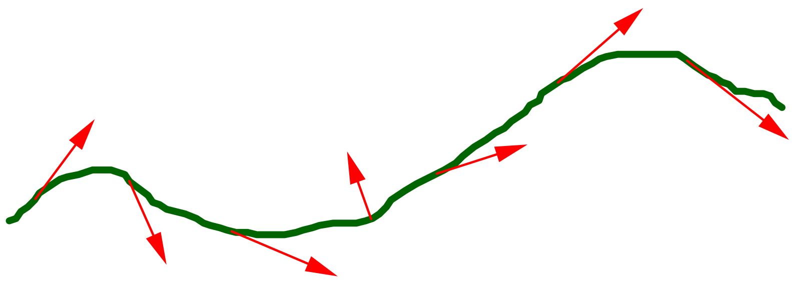


Figure 1: Line integral

Example 1. Find the work of the vector field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + \left(x - z^2\right)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, for $0 \leq t \leq 1$.

For points along the curve we have

$$x = t \qquad y = t^2 \qquad z = t^3 \tag{3}$$

Therefore, the values of $\mathbf{F}$ along the curve are

$$\mathbf{F}(t) = \left(t^2 - t^2\right)\mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k} = (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k} \tag{4}$$

The velocity is

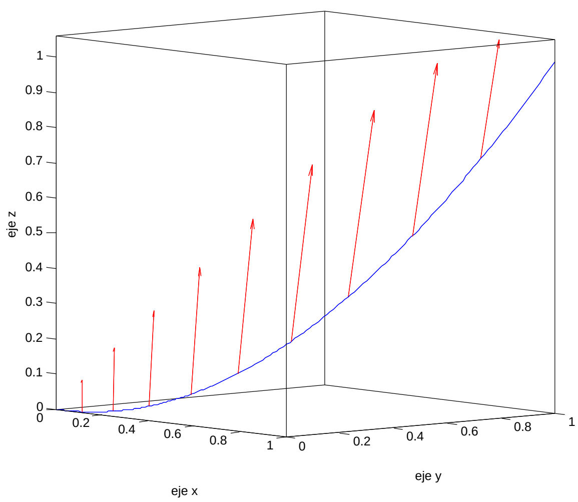
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \tag{5}$$

so

$$\mathbf{F} \cdot \mathbf{v} = \left((t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}\right) \cdot \left(\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}\right) = 2t^4 - 2t^5 + 3t^3 - 3t^8 \tag{6}$$

and the formula 2 says that the work is

$$\int_0^1 (\mathbf{F} \cdot \mathbf{v})\, dt = \int_0^1 \left(2t^4 - 2t^5 + 3t^3 - 3t^8\right) dt == \left(\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\right)\Big|_0^1 = \frac{29}{60} \tag{7}$$



Example 2. Find the work done on a particle moving from $(0,0)$ to $(3,3)$ along the trajectories C_1, C_2 shown below. Take $\mathbf{F} = (x^2y+1)\mathbf{i}$

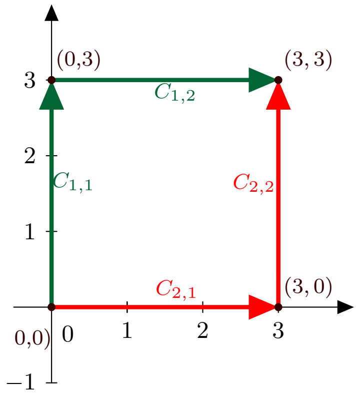


Figure 2: Line integral along different trajectories

We need to parameterize C_1, C_2. For C_1 we consider the segment from $(0,0)$ to $(0,3)$ (called $C_{1,1}$) and the second segment which goes from $(0,3)$ to $(3,3)$ (called $C_{1,2}$).

On $C_{1,1}$ we have $x = 0$ and we can take y as a parameter:

$$\mathbf{r}_{C_{1,1}}(y) = y\mathbf{j} \qquad 0 \le y \le 3 \tag{8}$$

On $C_{1,1}$ the value of $\mathbf{F}$

$$\mathbf{F} = \mathbf{i} \tag{9}$$

Since

$$\frac{d}{dy}\mathbf{r}_{C_{1,1}} = \mathbf{j} \tag{10}$$

The work done along $C_{1,1}$ is

$$\int_{C_{1,1}} \mathbf{F}\cdot d\mathbf{r} = \int_0^3 \mathbf{F}\cdot\frac{d}{dy}\mathbf{r}_{C_{1,1}}dy = \int_0^3 0dy = 0 \tag{11}$$

For $C_{1,2}$ we have $y = 3$ and we take x as a parameter, that is,

$$\mathbf{r}_{C_{1,2}}(x) = x\mathbf{i} + 3\mathbf{j} \qquad 0 \le x \le 3 \tag{12}$$

On $C_{1,2}$ the value of $\mathbf{F}$ is

$$\mathbf{F} = \left(3x^2 + 1\right)\mathbf{i} \tag{13}$$

Since

$$\frac{d}{dx}\mathbf{r}_{C_{1,2}} = \mathbf{i} \tag{14}$$

The work done along $C_{1,2}$ is

$$\int_{C_{1,2}} \mathbf{F}\cdot d\mathbf{r} = \int_0^3 \mathbf{F}\cdot\frac{d}{dx}\mathbf{r}_{C_{1,2}}dx = \int_0^3 \left(3x^2+1\right)dx = 30 \tag{15}$$

The total work along C_1 is the sum of these two works, i.e,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_{1,1}} \mathbf{F} \cdot d\mathbf{r} + \int_{C_{1,2}} \mathbf{F} \cdot d\mathbf{r} = 30 \tag{16}$$

For C_2 we consider the segment from $(0,0)$ to $(3,0)$ (called $C_{2,1}$) and the one from $(3,0)$ to $(3,3)$ (called $C_{2,2}$).

On $C_{2,1}$ we have $y = 0$ and we use x as parameter, that is,

$$\mathbf{r}_{C_{2,1}}(x) = x\mathbf{i} \qquad 0 \leq x \leq 3 \tag{17}$$

On $C_{2,1}$ the value of $\mathbf{F}$ is

$$\mathbf{F} = \mathbf{i} \tag{18}$$

Since

$$\frac{d}{dx}\mathbf{r}_{C_{2,1}} = \mathbf{i} \tag{19}$$

The work along $C_{2,1}$ is

$$\int_{C_{2,1}} \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \mathbf{F} \cdot \frac{d}{dx}\mathbf{r}_{C_{2,1}} dx = \int_0^3 1dx = 3 \tag{20}$$

On $C_{2,2}$ we have $x = 3$ and we can use y as a parameter

$$\mathbf{r}_{C_{2,2}}(y) = 3\mathbf{i} + y\mathbf{j} \qquad 0 \leq y \leq 3 \tag{21}$$

On $C_{2,2}$ the value of $\mathbf{F}$ is

$$\mathbf{F} = (9y+1)\,\mathbf{i} \tag{22}$$

Since

$$\frac{d}{dy}\mathbf{r}_{C_{2,2}} = \mathbf{j} \tag{23}$$

The work along the curve $C_{2,2}$ is

$$\int_{C_{2,2}} \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \mathbf{F} \cdot \frac{d}{dy}\mathbf{r}_{C_{2,2}} dy = 0 \tag{24}$$

The total work along C_2 is the sum of the works, that is

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_{2,1}} \mathbf{F} \cdot d\mathbf{r} + \int_{C_{2,2}} \mathbf{F} \cdot d\mathbf{r} = 3 \tag{25}$$

In particular, **notice that the work depended on which trajectory we chose!**

Conservative vector field:

- A vector field $\mathbf{F}$ is a **conservative vector field** if its line integral is the same for all curves connecting two arbitrary points P, Q. That is, the work is trajectory independent. Equivalently, the line integral of $\mathbf{F}$ along any loop must vanish, which we write as

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \tag{26}$$

- A scalar field V is called a **potential function** for the vector field $\mathbf{F}$ if

$$\mathbf{F} = \nabla V \tag{27}$$

(the physicists would write this as $\mathbf{F} = -\nabla V$)

- In the case $\mathbf{F}$ can be obtained from a potential function V as above then $\mathbf{F}$ will be automatically a conservative vector field and moreover the line integral is simply

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = V(Q) - V(P) \tag{28}$$

- Whenever the region we are studying is simply connected (every loop can be contracted to a point within this region), a vector field $\mathbf{F}$ will be conservative if and only if such a potential function V can be found. Notice that V will not be unique, but rather different potential functions differ by a constant. In order to find a candidate V, we must solve the equations

$$\frac{\partial V}{\partial x} = F_1 \qquad \frac{\partial V}{\partial y} = F_2 \qquad \frac{\partial V}{\partial z} = F_3 \tag{29}$$

where we wrote $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

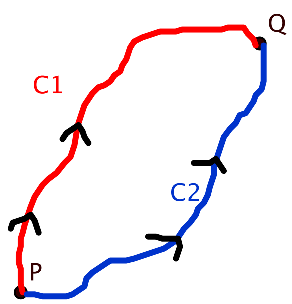


Figure 3: Conservative Vector Field

Example 3. The vector field $\mathbf{F} = \left(x + y^2\right)\mathbf{i} + \left(2xy + 3y^2\right)\mathbf{j} + \mathbf{k}$ is conservative. Find the potential functions.

We must solve

$$\underbrace{\frac{\partial V}{\partial x} = x + y^2}_{(1)} \quad \underbrace{\frac{\partial V}{\partial y} = 2xy + 3y^2}_{(2)} \quad \underbrace{\frac{\partial V}{\partial z} = 1}_{(3)} \tag{30}$$

Integrating (1) with respect to x we obtain

$$V = \frac{x^2}{2} + y^2 x + f(y, z) \tag{31}$$

where instead of a number c as a constant of integration we use an arbitrary function $f(y, z)$ since we treated y, z as constants. Differentiating this formula for V with respect to y we have

$$\frac{\partial V}{\partial y} = 2yx + \frac{\partial f(y, z)}{\partial y} \tag{32}$$

comparing with (2) we conclude that

$$\frac{\partial f(y, z)}{\partial y} = 3y^2 \tag{33}$$

Integrating with respect to y we have

$$f(y, z) = y^3 + g(z) \tag{34}$$

where $g(z)$ is our "constant" of integration. In this way

$$V = \frac{x^2}{2} + xy^2 + y^3 + g(z) \tag{35}$$

comparing with (3) we obtain that

$$\frac{dg}{dz} = 1 \tag{36}$$

Integrating with respect to z we find

$$g = z + c \tag{37}$$

where c is truly a constant. In this way the potential functions are

$$V = \frac{x^2}{2} + xy^2 + y^3 + z + c \tag{38}$$

Example 4. Determine whether $\mathbf{F} = (2x - 3)\,\mathbf{i} - z\mathbf{j} + \cos z\mathbf{k}$ is a conservative vector field or not.

If $\mathbf{F}$ were conservative then 29 would have a solution. So we want to solve

$$\underbrace{\frac{\partial V}{\partial x} = 2x - 3}_{(1)} \quad \underbrace{\frac{\partial V}{\partial y} = -z}_{(2)} \quad \underbrace{\frac{\partial V}{\partial z} = \cos z}_{(3)} \tag{39}$$

Integrating (1) we obtain

$$V = x^2 - 3x + f(y,z) \tag{40}$$

Therefore

$$\frac{\partial V}{\partial y} = \frac{\partial f(y,z)}{\partial y} \tag{41}$$

comparing with (2) we obtain

$$\frac{\partial f(y,z)}{\partial y} = -z \tag{42}$$

Integrating with respect to y we obtain

$$f(y,z) = -yz + g(z) \tag{43}$$

Therefore

$$V = x^2 - 3x - yz + g(z) \tag{44}$$

Differentiating with respect to z

$$\frac{\partial V}{\partial z} = -y + \frac{\partial g}{\partial z} \tag{45}$$

and comparing with (3) we obtain

$$\frac{\partial g(z)}{\partial z} - \cos z = y \tag{46}$$

The last equation cannot be satisfied all the time, since the left hand side depends on z while the right hand side depends on y. Therefore, the vector field is not conservative.

On a simply connected region, a vector field **F** is conservative if and only if

$$\frac{\partial}{\partial y}F_1 = \frac{\partial}{\partial x}F_2 \quad \frac{\partial}{\partial y}F_3 = \frac{\partial}{\partial z}F_2 \quad \frac{\partial}{\partial x}F_3 = \frac{\partial}{\partial z}F_1 \tag{47}$$

When $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is defined on the xy plane we just need to check the condition

$$\frac{\partial}{\partial y}F_1 = \frac{\partial}{\partial x}F_2 \tag{48}$$

Example 5. Evaluate the line integral $\int_C ydx + xdy + 4dz$ where C is the line segment from $(1,1,1)$ to $(2,3,-1)$.

The vector field is

$$\mathbf{F} = y\mathbf{i} + x\mathbf{j} + 4\mathbf{k} \tag{49}$$

Since

$$\frac{\partial}{\partial y}F_1 = 1 = \frac{\partial}{\partial x}F_2 \tag{50}$$

$$\frac{\partial}{\partial y}F_3 = 0 = \frac{\partial}{\partial z}F_2 \tag{51}$$

$$\frac{\partial}{\partial x}F_3 = 0 = \frac{\partial}{\partial z}F_1 \tag{52}$$

the vector field is conservative. To find V we must solve

$$\underbrace{\frac{\partial V}{\partial x} = y}_{(1)} \quad \underbrace{\frac{\partial V}{\partial y} = x}_{(2)} \quad \underbrace{\frac{\partial V}{\partial z} = 4}_{(3)} \tag{53}$$

Integrating (3) with respect to z we find

$$V = 4z + f(x,y) \tag{54}$$

Differentiating with respect to y we have

$$\frac{\partial V}{\partial y} = \frac{\partial f(x,y)}{\partial y} \tag{55}$$

and comparing with (2) we have

$$\frac{\partial f(x,y)}{\partial y} = x \tag{56}$$

integrating with respect to y we find

$$f(x,y) = xy + g(x) \tag{57}$$

Therefore

$$V = 4z + xy + g(x) \tag{58}$$

differentiating with respect to x

$$\frac{\partial V}{\partial x} = y + \frac{\partial g}{\partial x} \tag{59}$$

and comparing with (1) we obtain that

$$g(x) = c \tag{60}$$

is a constant so the potential is

$$V(x,y,z) = 4z + xy + c \tag{61}$$

Therefore the line integral can be obtained by evaluating the potential at the endpoints

$$\int_C ydx + xdy + 4dz = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C dV = V(2,3,-1) - V(1,1,1) = -3 \tag{62}$$

Notice that the final answer is insensitive to the constant of integration used.

Field Lines

A curve $\mathbf{r}(t)$ is a **field line** of a vector field $\mathbf{F}(x,y,z)$ if at every time t, the tangent vector $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ agrees with $\mathbf{F}$ at that point, that is,

$$\mathbf{v}(t) = \mathbf{F}(\mathbf{r}(t)) \tag{63}$$

Example 6. Find the field lines of the vector field $\mathbf{F} = -x\mathbf{i} + y\mathbf{j}$.

We must solve the system

$$\frac{dx}{dt} = -x \qquad \frac{dy}{dt} = y \tag{64}$$

Notice that we can "solve" for dt as follows:

$$dt = -\frac{dx}{x} = \frac{dy}{y} \tag{65}$$

Working with the last two equations we obtain

$$-\frac{dx}{x} = \frac{dy}{y} \tag{66}$$

We integrate each side individually and find

$$-\ln x = \ln y + c \tag{67}$$

In other words

$$xy = C \tag{68}$$

where C is a new constant.

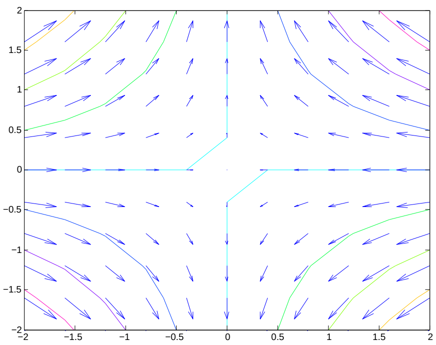


Figure 4: Vector field lines

If $\mathbf{F}$ is a conservative vector field then,

$$\mathbf{F} = \nabla V \tag{69}$$

Now, the potential function evaluated along a field line $\mathbf{r}(t)$ can be regarded as a function of time

$$V(t) = V(\mathbf{r}(t)) \tag{70}$$

The chain rule then says that

$$\frac{dV}{dt} = (\nabla V)(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{F}(\mathbf{r}(t)) = \|\mathbf{F}(\mathbf{r}(t))\|^2 \tag{71}$$

Since the right hand side is never negative we conclude that

$$\frac{dV}{dt} \geq 0 \quad \text{along a field line} \tag{72}$$

In other words, the potential is always non-decreasing along a field lines. In particular, unless $\mathbf{F}$ vanishes at every point along a field line, there cannot be any field lines for a conservative vector field which are closed loops since V would necessarily have to decrease as you travel back.

Moreover, recall that for an equipotential surface

$$V = \text{constant} \tag{73}$$

The gradient vector field ∇V gives the normal vector to these surfaces. Now, $\nabla V = \mathbf{F}$ is precisely the velocity vector of a field line, which means that field lines are perpendicular to the equipotential surfaces!

- **If a conservative vector field does not vanish at every point of a field line then the field line cannot be closed loops.**
- **For a conservative field, the field lines are perpendicular to the equipotential surfaces.**

For example, the electric field produced by a particle with charge q centered at the origin is

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{e}_r}{r^2} \tag{74}$$

where $\mathbf{e}_r$ is the unitary vector in spherical coordinates and $r = |\mathbf{r}|$. It is a good exercise to verify that

$$\mathbf{E} = -\nabla \left(\frac{q}{4\pi\varepsilon_0} \frac{1}{r} \right) \tag{75}$$

so the electric field is conservative. Our previous discussion implies that:

1. The electric field lines are never closed loops.
2. Since the equipotential surfaces are sphere centered at the origin, the electric field lines are rays emanating from the origin.

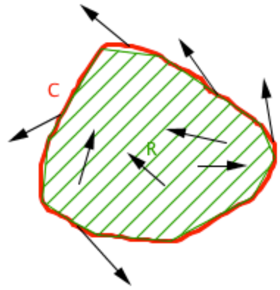


Figure 5: Green's Theorem in the plane

Now we will discuss one of the major theorems of vector calculus. We will consider a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$, which we will think as a velocity vector field, so that at the point (x, y) of some region R, $\mathbf{F}(x, y)$ represents the velocity of a fluid. The only particles of the fluid that can escape the region R must go through the curve C.

We can break the velocity of those particles into a tangent component and a normal component to the curve. Clearly only the normal component is responsible for the escape of the particles, while the tangential component is related to the vorticity of the fluid.

We will analyze first such a vorticity. For doing this we compute

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds \tag{76}$$

where $\oint$ means our curve is closed. Using the definition of the tangent vector we have that

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} ds = \oint_C \mathbf{F} \cdot d\mathbf{r} \tag{77}$$

which means that computing the vorticity of $\mathbf{F}$ is equivalent to compute the work of $\mathbf{F}$ along the curve C.

To find Green's theorem, we analyze first the case of a very small rectangle centered at the point (x, y) with sides of length $2\triangle x$ and $2\triangle y$. Clearly

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} \tag{78}$$

where we are using the counter clockwise direction. Each curve is parameterized as

$$\begin{cases} C_1 & \mathbf{r}(t) = t\mathbf{i} + (y - \triangle y)\,\mathbf{j} & x - \triangle x \leq t \leq x + \triangle x \\ C_2 & \mathbf{r}(t) = (x + \triangle x)\mathbf{i} + t\mathbf{j} & y - \triangle y \leq t \leq y + \triangle y \\ C_3 & \mathbf{r}(t) = t\mathbf{i} + (y + \triangle y)\,\mathbf{j} & x + \triangle x \leq t \leq x - \triangle x \\ C_4 & \mathbf{r}(t) = (x - \triangle x)\mathbf{i} + t\mathbf{j} & y + \triangle y \leq t \leq y - \triangle y \end{cases} \tag{79}$$

so the computation of the work becomes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{x-\triangle x}^{x+\triangle x} F_1 dt + \int_{y-\triangle y}^{y+\triangle y} F_2 dt + \int_{x+\triangle x}^{x-\triangle x} F_1 dt + \int_{y+\triangle y}^{y-\triangle y} F_2 dt \tag{80}$$

We will approximate each integral using the value the vector fields takes in the middle of the trajectory, namely

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} \simeq F_1(x,y-\triangle y)(2\triangle x) + F_2(x+\triangle x,y)\,(2\triangle y)\\ -F_1(x,y+\triangle y)\,(2\triangle x) - F_2(x-\triangle x,y)(2\triangle y)\end{aligned} \tag{81}$$

Rearranging things we find that

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} \simeq (F_1(x,y-\triangle y) - F_1(x,y+\triangle y))\,(2\triangle x)\\ +\,(F_2(x+\triangle x,y) - F_2(x-\triangle x,y))\,(2\triangle y)\end{aligned} \tag{82}$$

Finally, we do a Taylor expansion to first order about (x,y) to obtain

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \simeq 4\left[-\frac{\partial F_1\,(x,y)}{\partial y} + \frac{\partial F_2(x,y)}{\partial x}\right]\triangle x\triangle y \tag{83}$$

Since the are of the rectangle $4\triangle x\triangle y$ this means that the vorticity per unit are is

$$\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1\,(x,y)}{\partial y} = \lim_{\triangle A\longrightarrow 0}\frac{\oint_C \mathbf{F}\cdot d\mathbf{r}}{\triangle A} \tag{84}$$

The same calculations can be done using the xz or yz planes, so we just found

Curl of a vector field:

If **F** is a vector field, the **curl** of **F** is the amount of vorticity per unit area. It is denoted $\nabla\times\mathbf{F}$ and defined as

$$\hat{\mathbf{n}}\cdot(\nabla\times\mathbf{F}) = \lim_{\triangle S\longrightarrow 0}\frac{\oint_C \mathbf{F}\cdot d\mathbf{r}}{\triangle S} \tag{85}$$

Here $\hat{\mathbf{n}}$ is a unitary vector normal to the surface $\triangle S$ and C is the curve which is the boundary of the surface $\triangle S$. We move along the curve using the right hand rule: if your thumb points gives you the direction of motion along curve then as you close your hand the remaining fingers point towards the vector $\hat{\mathbf{n}}$.

In cartesian coordinates the curl can be computing using the "determinant"

$$\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \tag{86}$$

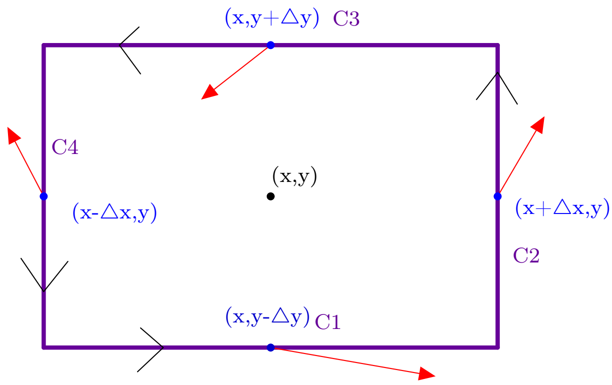


Figure 6: Curl over a small rectangle

Example 7. Compute the curls of $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ and $\mathbf{G} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$

From the figures shown below it looks like both vector fields are spinning so one might assume that the curl of both fields will be non-zero in both situations. However, notice that

$$\nabla \times \mathbf{F} = \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) \mathbf{k} = 2\mathbf{k} \tag{87}$$

while

$$\nabla \times \mathbf{G} = \left(\frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right) - \frac{\partial}{\partial y} \left(\frac{-y}{x^2+y^2} \right) \right) \mathbf{k} = \frac{(x^2+y^2) - 2x^2 + (x^2+y^2) - 2y^2}{(x^2+y^2)^2} \mathbf{k} = \mathbf{0} \tag{88}$$

for any (x, y) which is not the origin for the last vector field. So what is the difference between both vector fields?

The important thing to remember is that the curl indicates the infinitesimal rotation near a point, that is, it computes whether there is a net rotation or not at each point.

To compute $\nabla \times \mathbf{F}$ at the point (x, y, z) we place a small paddle wheel at the point (x, y, z) and we determine whether it starts rotating about an axis. If a point P on the paddle wheel starts rotating with angular velocity $\boldsymbol{\omega}$ then its velocity is (here we choose coordinates so that P is the origin)

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{89}$$

Therefore, in this case

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix} = 2\boldsymbol{\omega} \tag{90}$$

Hence we can think of the curl of a vector field as twice the angular velocity of a paddle wheel.

Returning to our vector fields $\mathbf{F}$ and $\mathbf{G}$, notice that $\|\mathbf{F}\| = \sqrt{x^2+y^2}$ while $\mathbf{G} =$

$\frac{1}{\sqrt{x^2+y^2}}$. Therefore, the torque on the paddle wheel produced by $\mathbf{F}$ is about (ignoring the sine of the angle for our analysis) $\|\mathbf{F}\|\|\mathbf{r}\| = x^2 + y^2$, while the torque induced by $\mathbf{G}$ is about $\|\mathbf{G}\|\|\mathbf{r}\| = 1$. Therefore, we see that in the first case the torque does change with the position (x, y), which is why the paddle wheel can start rotation, while in the second case the torque at different points has about the same magnitude, which is why they cancel each other out and the paddle wheel will not try to rotate.

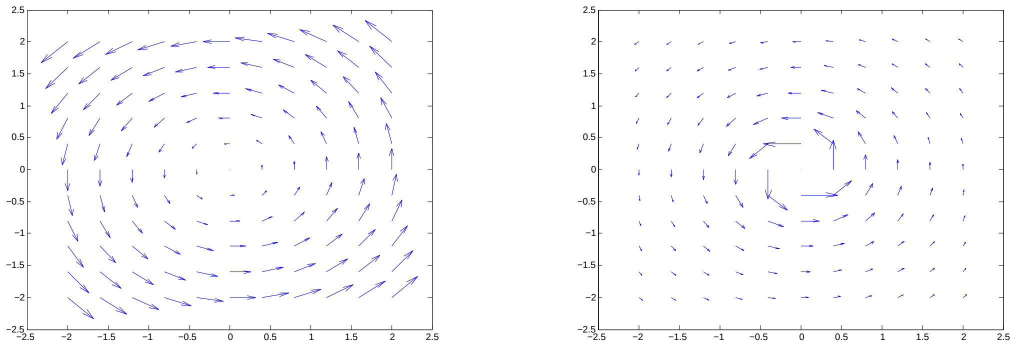


Figure 7: Vector fields $\mathbf{F}$ and $\mathbf{G}$

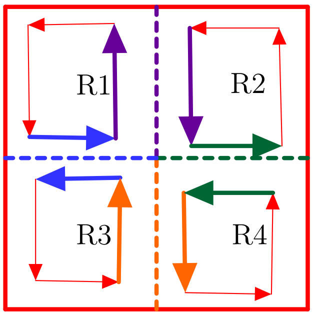
Now that we know how to compute the curl, we can return to our problem of computing the quantity

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \tag{91}$$

We already know that on sufficiently small rectangles with sides $\triangle x, \triangle y$

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int\int (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \tag{92}$$

where $\mathbf{n} = \mathbf{k}$ and dS is the area differential. If our rectangle is not small we can break it into smaller rectangles. If C_1, C_2, C_3, C_4, C are the boundary curves of R_1, R_2, R_3, R_4, R then



$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_{C_1} \mathbf{F}\cdot d\mathbf{r} + \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} + \oint_{C_3} \mathbf{F}\cdot d\mathbf{r} + \oint_{C_4} \mathbf{F}\cdot d\mathbf{r} \tag{93}$$

since the internal contributions cancel each other out since they are travelled twice in opposite directions. Assuming the sub-rectangles are small enough we can use 92 and conclude that

$$\begin{gathered}\oint_C \mathbf{F}\cdot d\mathbf{r} = \\ \int\int_{R_1} (\nabla\times\mathbf{F})\cdot\mathbf{n}dS + \int\int_{R_2} (\nabla\times\mathbf{F})\cdot\mathbf{n}dS + \int\int_{R_3} (\nabla\times\mathbf{F})\cdot\mathbf{n}dS + \int\int_{R_4} (\nabla\times\mathbf{F})\cdot\mathbf{n}dS \\ = \int\int_R (\nabla\times\mathbf{F})\cdot\mathbf{n}dS\end{gathered} \tag{94}$$

An arbitrary region can then be approximated by rectangles so we just found Green's theorem!

Green's Theorem: tangential version

Let R be a region on the xy plane which is bounded and simply connected. If $\mathbf{F}$ is a differentiable vector field defined everywhere on R and C is the boundary curve of R which we assume to be closed without self intersections, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_R (\nabla\times\mathbf{F})\cdot\mathbf{n}dS \tag{95}$$

Letting $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$ we can write this as

$$\oint_C F_1dx + F_2dy = \int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy \tag{96}$$

where **we travel along the curve C in such a way that R is to our left as we move along C.**

Example 8. Verify Green's theorem for the rectangle $\mathbf{F}(x,y) = (x-y)\,\mathbf{i} + x\mathbf{j}$ and using the region R bounded by the circle $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le 2\pi$

We will compute both sides of Green's theorem and check that they agree. For the left hand side we use the parameterization of the circle given to us.

Since $x = \cos t$ we have $dx = -\sin t dt$ and for $y = \sin t$ we have $dy = \cos t dt$. Thus

$$\oint F_1 dx + F_2 dy = \int_0^{2\pi} \left((\cos t - \sin t)(-\sin t) + \cos t (\cos t) \right) dt = \int_0^{2\pi} (1 - \cos t \sin t)\, dt = 2\pi \tag{97}$$

The right hand side of 96 can be computed using polar coordinates

$$\int \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \int_0^{2\pi} \int_0^1 (1+1)\, \rho d\rho d\varphi = 2\pi \tag{98}$$

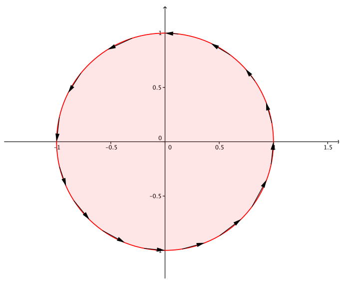
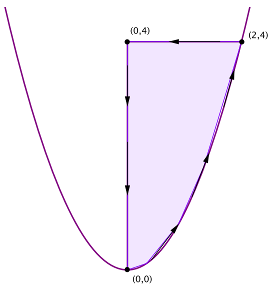


Figure 8: Green's theorem on the circle

Example 9. Use Green's theorem to compute $\oint \left(x^2 + y^2\right) dx + xydy$ where the curve has three pieces: the part of the parabola $y = x^2$ from $(0, 0)$ to $A = (2, 4)$,the segment from A to $B = (0, 4)$, and the segment from B to the origin O.

Thanks to Green's theorem, rather than computing three line integrals, we just find



$$\int_0^4 \left(\int_0^{\sqrt{y}} (y - 2y)\, dx \right) dy = -\int_0^4 yx|_0^{\sqrt{y}} dy = -\int_0^4 y^{\frac{3}{2}} dy = -\frac{2}{5} (4)^{\frac{5}{2}} = -\frac{64}{5} \tag{99}$$

Green's Theorem can also be used to compute the area of a region R. Just take the vector field $\mathbf{F} = -\frac{1}{2}y\mathbf{i} + \frac{1}{2}x\mathbf{j}$ en 96 in order to obtain

$$\frac{1}{2}\oint -ydx + xdy = \int\int_R dxdy = A \tag{100}$$

The area of a region R can be computed as

$$A = \int\int_R dxdy = \frac{1}{2}\oint -ydx + xdy \tag{101}$$

One can also use the formulas

$$A = \oint xdy = -\oint ydx \tag{102}$$

Example 10. Use Green's theorem to compute the area of the ellipse $\mathbf{r}(t) = a\cos t\mathbf{i} + b\sin t\mathbf{j}$ with $0 \le t < 2\pi$.

Taking $x = a\cos t$ and $y = b\sin t$ we have $dy = b\cos tdt$ so using the second formula for the area $\oint xdy$ stated above, we need to compute

$$A = \int_0^{2\pi} (a\cos t)\,(b\cos t)\,dt = \pi ab \tag{103}$$

In the case the region R is not simply connected, it is possible to adjust Green's Theorem.

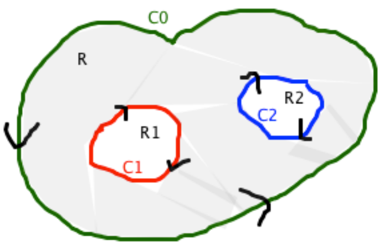


Figure 9: Green's theorem on a region with holes

For example, to apply Green's theorem to the region R shown above we work with $R^* = R \cup R_1 \cup R_2$, which is simply connected. Clearly

$$\begin{gathered}\int\int_R (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS = \\ \int\int_{R^*} (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS - \int\int_{R_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS - \int\int_{R_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS\end{gathered} \tag{104}$$

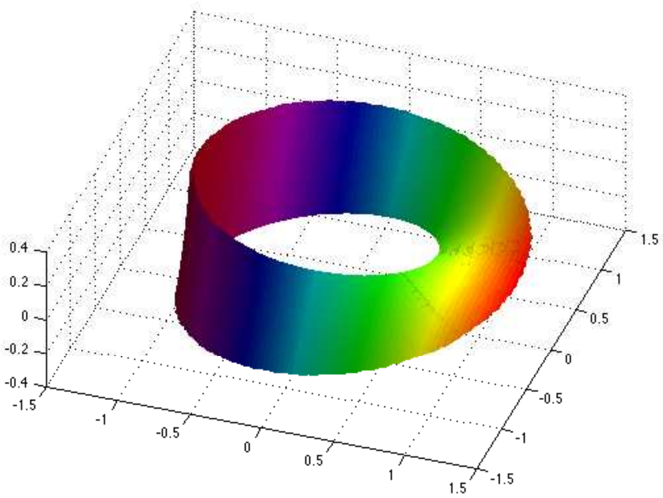
and since each of the regions R^*, R_1, R_2 is simply connected we can apply Green's theorem here and we find that

$$\begin{aligned} \int\int_R (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \int_{C_0} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{C_0 \cup C_1 \cup C_2} \mathbf{F} \cdot d\mathbf{r} \end{aligned} \tag{105}$$

where we traveling the curve as shown in the figure.

Stoke's theorem can be considered as the generalization of Green's Theorem in the case the region is not on the plane, and rather represents a surface in space. In this case we need to specify what the normal vector is supposed to be. The two natural candidates are $\mathbf{n} = \pm\frac{\nabla f}{|\nabla f|}$, assuming the surface was defined by the equation $f(x, y, z) = 0$.

On the other hand, if we parameterize the surface in terms of u, v then we can use $\mathbf{n} = \pm\frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|}$. However, there are cases where the parameterization only works for certain regions of the surface and where we need to check if we can choose a normal vector in a consistent way. In way, there are one sided surfaces like the Mobius strip where this is not possible, although they need not concern us in this class.



Stokes' Theorem and Conservative Vector Fields

A surface S is **orientable** if it has a nowhere vanishing vector field $\mathbf{n}$. If S is defined as $f(x, y, z) = 0$, we can take

$$\mathbf{n} = \pm \frac{\nabla f}{|\nabla f|} \tag{106}$$

or

$$\mathbf{n} = \pm \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|} \tag{107}$$

whenever the surface is parameterized by u, v.

Stokes' theorem states that for an orientable surface S which is simply connected with boundary C

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \tag{108}$$

where the curve is oriented using the right hand rule.

Notice that if $\mathbf{F}$ is a conservative vector field then the left hand side of Stokes' theorem vanishes, regardless of the curve chosen. The only way for this to occur is if $\nabla \times \mathbf{F}$ vanishes as well. In other words:

A vector field F defined on a simply connected region is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$.

Example 11. Using Stokes' theorem, find $I = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + (y + z)\,\mathbf{k}$ and S is the portion of the surface $2x + y + z = 2$ above the first octant and $\mathbf{n}$ is the unitary normal vector to the surface, with non-negative z component.

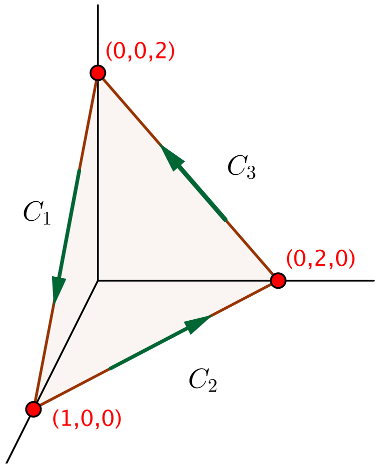
We can take

$$\mathbf{n} = \frac{2}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k} \tag{109}$$

Hence by Stokes' theorem

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \int \mathbf{F} \cdot d\mathbf{r} \tag{110}$$

where the line integral is taken as shown.



The curve C_1 is the intersection of the plane $2x + y + z = 2$ with $y = 0$, that is, $2x + z = 2$. The parameterization is $(x, 0, 2 - 2x)$ with $0 \leq x \leq 1$ so

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (x\mathbf{j} + (2 - 2x)\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{k})\, dx = \int_0^1 -4 + 4x dx = -2 \tag{111}$$

The curve C_2 is the intersection of the plane $2x + y + z = 2$ with $z = 0$, that is, $2x + y = 2$. The parameterization is $(x, 2 - 2x, 0)$ with $0 \leq x \leq 1$ and taking into account the orientation we are using we find that

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\int_0^1 ((2 - 2x)\mathbf{i} + x\mathbf{j} + (2 - 2x)\mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j})\, dx = \int_0^1 4x - 2dx = 0 \tag{112}$$

The curve C_3 is the intersection of the plane $2x + y + z = 2$ with $x = 0$, that is, $y + z = 2$. The parameterization is $(0, 2 - z, z)$ with $0 \leq z \leq 2$ so

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 ((2 - z)\mathbf{i} + 2\mathbf{k}) \cdot (-\mathbf{j} + \mathbf{k})\, dz = 4 \tag{113}$$

From this we conclude that

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 2 \tag{114}$$

Example 12. Use Stokes' theorem to evaluate $\int_C -y^3 dx + x^3 dy - z^3 dz$ where C is the intersection of the cylinder $x^2 + y^2 = 1$, and the plane $x + y + z = 1$. Assume C is oriented counterclockwise with respect to the xy plane.

Define

$$\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k} \tag{115}$$

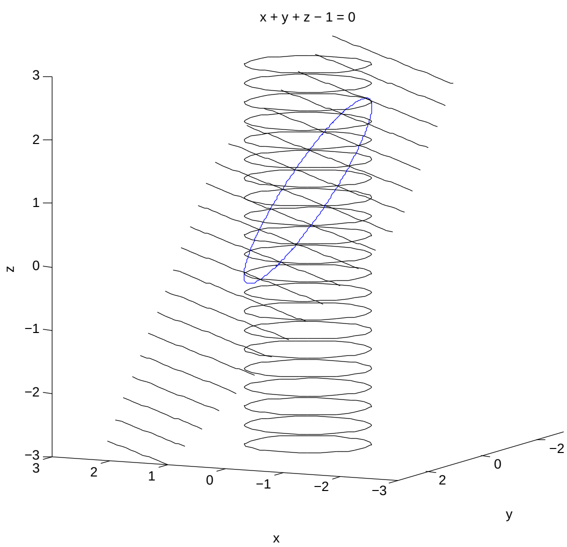
It is easy to find that

$$\nabla \times \mathbf{F} = 3\left(x^2 + y^2\right)\mathbf{k} \tag{116}$$

On the other hand, we write the normal vector as $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ so that

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 3\left(x^2 + y^2\right) n_3 \tag{117}$$

From the figure



we can take our surface to be the disc whose boundary is the ellipse shown in blue. Since the disc belongs to the plane $x + y + z = 1$ we can take the normal vector of the plane as the normal vector for the disc and use

$$\mathbf{n} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{j} \tag{118}$$

Equivalently, we could have parameterized the disc using x, y as

$$\mathbf{r}(x, y) = (x, y, 1 - x - y) \tag{119}$$

so that

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1) \tag{120}$$

In any case,

$$n_3 = \frac{1}{\sqrt{3}} \tag{121}$$

and

$$dS = \sqrt{3}dxdy \tag{122}$$

Since $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 3\left(x^2 + y^2\right) n_3$ we obtain

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n}dS = 3\left(x^2 + y^2\right) dxdy \tag{123}$$

and thanks to Stokes' theorem we conclude that

$$\int_C -y^3 dx + x^3 dy - z^3 dz = 3 \int\int \left(x^2 + y^2\right) dxdy \tag{124}$$

To integrate the right hand side we use polar coordinates

$$3 \int_0^{2\pi} \int_0^1 \left(r^2\right) r dr d\theta = \frac{6\pi}{4} \tag{125}$$

Example 13. Verify Stokes' theorem for the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$, the its boundary $x^2 + y^2 = 9, z = 0$ and the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

First of all a parameterization for the circle is $\mathbf{r}(\varphi) = 3\cos\varphi\mathbf{i}+3\sin\varphi\mathbf{j}$ with $0 \leq \varphi \leq 2\pi$ so the work is

$$\int_0^{2\pi} (3\sin\varphi\mathbf{i} - 3\cos\varphi\mathbf{j}) \cdot (-3\sin\varphi\mathbf{i} + 3\cos\varphi) d\varphi = -18\pi \tag{126}$$

We can find that $\nabla \times \mathbf{F} = -2\mathbf{k}$ and to parameterize the hemisphere we use spherical coordinates,

$$\mathbf{r}(\theta, \varphi) = 3\sin\theta\cos\varphi\mathbf{i} + 3\sin\theta\sin\varphi\mathbf{j} + 3\cos\theta\mathbf{k} \tag{127}$$

A normal vector to the surface is

$$\mathbf{r}_\theta \times \mathbf{r}_\varphi = 9 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{vmatrix} = 9\left(\sin^2\theta\cos\varphi, \sin^2\theta\sin\varphi, \cos\theta\sin\theta\right) \tag{128}$$

which we normalize as

$$\mathbf{n} = \frac{\mathbf{r}_\theta \times \mathbf{r}_\varphi}{|\mathbf{r}_\theta \times \mathbf{r}_\varphi|} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{129}$$

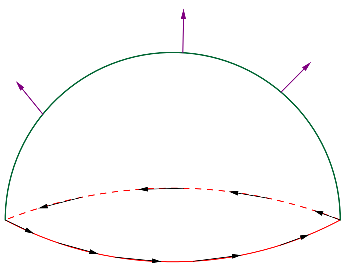
Thus the surface differential is

$$dS = 9\sin\theta d\theta d\varphi \tag{130}$$

and we must integrate

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\frac{\pi}{2}}(0,0,-2)\cdot(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)\,9\sin\theta d\theta d\varphi \\
=&-18\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta d\theta d\varphi \\
=&-18\pi\int_0^{\frac{\pi}{2}}\sin(2\theta)\,d\theta \\
=&-18\pi
\end{aligned}$$

as desired.



Now we return to our model of the fluid escaping a region R. As we mentioned before, the normal component to the velocity is the one responsible for the escape of fluid through the boundary of the surface.

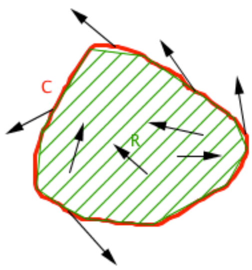


Figure 10: Green's theorem on the plane

In fact, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ and $\mathbf{T} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}$ is the tangent vector the it is easy to check that [1]

$$\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j} \tag{131}$$

is the desired vector (given our orientation conventions).

[1] In fact, $\mathbf{n} = -\mathbf{N}$ where $\mathbf{N}$ was the normal vector we defined when we studied curve.

Flux
If C is a curve on the plane and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is a vector field the **flux** of $\mathbf{F}$ along the curve C is

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \oint_C F_1 dy - F_2 dx \tag{132}$$

As before, to find the flux we start analyzing a very small rectangle centered at (x, y) with sides $\triangle x, \triangle y$.

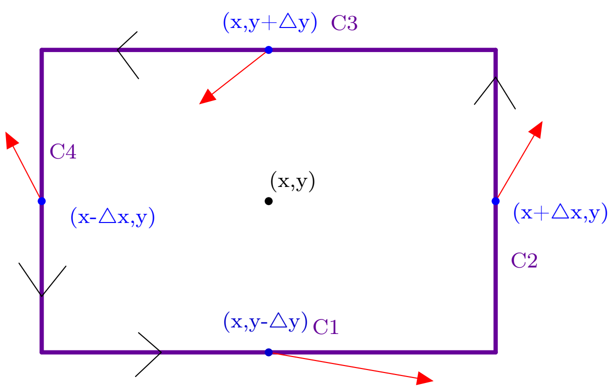


Figure 11: Flux on a rectangle

Clearly

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \int_{C_1} \mathbf{F} \cdot \mathbf{n} ds + \int_{C_2} \mathbf{F} \cdot \mathbf{n} ds + \int_{C_3} \mathbf{F} \cdot \mathbf{n} ds + \int_{C_4} \mathbf{F} \cdot \mathbf{n} ds \tag{133}$$

where we are orienting things counter clockwise. Again, we use the mid point approximation for the values of the vector field along each curve, which means

$$\begin{cases} \int_{C_1} \mathbf{F} \cdot \mathbf{n} ds \simeq -F_2(x, y - \triangle y)\,(2\triangle x) \\ \int_{C_2} \mathbf{F} \cdot \mathbf{n} ds \simeq F_1(x + \triangle x, y)\,(2\triangle y) \\ \int_{C_3} \mathbf{F} \cdot \mathbf{n} ds \simeq F_2(x, y + \triangle y)\,(2\triangle x) \\ \int_{C_4} \mathbf{F} \cdot \mathbf{n} ds \simeq -F_1(x - \triangle x, y)\,(2\triangle y) \end{cases} \tag{134}$$

Adding all these contributions and doing a Taylor expansion to first order

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds \simeq 4 \left(\frac{\partial F_1(x, y)}{\partial x} + \frac{\partial F_2(x, y)}{\partial y} \right) \triangle x \triangle y \tag{135}$$

Since the area of the rectangle is $4\triangle x\triangle y$ we can use the previous computation to define the **divergence** $\nabla \cdot \mathbf{F}$ of $\mathbf{F}$ as the flux per unit area

$$\nabla \cdot \mathbf{F} \equiv \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \tag{136}$$

As in the case for the vorticity, when the region is not a small rectangle we can break it into such pieces and the internal contributions will cancel. Therefore we found the second version of Green's theorem.

Green's Theorem: Normal Form

Let R be a region satisfying the same conditions as in the statement of the tangential form of Green's theorem. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is a vector field defined on R then

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \int\int_R \nabla \cdot \mathbf{F} dx dy \tag{137}$$

where $\mathbf{n}$ is a normal vector to the curve which points outward and the orientation of the curve is such that the region is to the left of our motion. It can also be written as

$$\oint_C F_1 dy - F_2 dx = \int\int_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dx dy \tag{138}$$

Example 14. Find the divergence of $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and $\mathbf{G} = \frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}$

In this case the vector fields look like

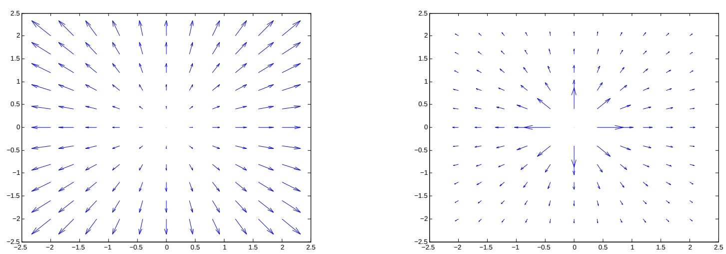


Figure 12: Vector fields **F** and **G**

Recall that at each point (x, y), $\nabla \cdot \mathbf{F}$ measure whether there is a net loss or gain of mass (or fluid). It is easy to compute that

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) = 2 \tag{139}$$

which can be interpreted as saying that each point is a net source of fluid.

On the other hand,

$$\nabla \cdot \mathbf{G} = \left(\frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2+y^2} \right) \right) = \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{\left(x^2+y^2\right)^2} = 0 \tag{140}$$

which again means that except at the origin (where nothing is defined), there is no net loss or gain of mass (or fluid).

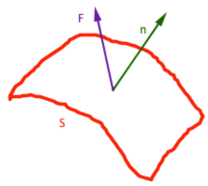


Figure 13: Flux through a surface

Divergence Theorem

The divergence theorem generalizes Green's normal theorem to surfaces in space. Assuming the surface S is orientable, we define the **flux** Φ through the surface as

$$\Phi = \int \int_S \mathbf{F} \cdot \mathbf{n} dS \tag{141}$$

The **divergence** $\nabla \cdot \mathbf{F}$ of a vector field is the density of the flux per unit volume

$$\nabla \cdot \mathbf{F}(x, y, z) = \lim_{\triangle V \longrightarrow 0} \frac{\int \int_S \mathbf{F} \cdot \mathbf{n} dS}{\triangle V} \tag{142}$$

where $\triangle V$ is the volume of a small region containing (x, y, z) while S is the boundary of $\triangle V$. The divergence is a scalar field and if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ then

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{143}$$

- If $\nabla \cdot \mathbf{F}(x, y, z) > 0$ ($\nabla \cdot \mathbf{F}(x, y, z) < 0$) then (x, y, z) is called a **source** (**sink**).
- If $\nabla \cdot \mathbf{F} = 0$ at every point (x, y, z), we say that $\mathbf{F}$ is a solenoidal field.

Gauss' theorem states that if R is a simply connected 3d region and its boundary S is closed without self intersections then

$$\int \int_S \mathbf{F} \cdot \mathbf{n} dS = \int \int \int_{\mathcal{R}} \nabla \cdot \mathbf{F} dV \tag{144}$$

Example 15. For $\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S the unit sphere $x^2 + y^2 + z^2 = 1$, compute $\int \mathbf{F} \cdot \mathbf{n} dS$

Using Gauss' theorem and the fact that $\nabla \cdot \mathbf{F} = 2 + 2y + 2z$ we need to compute

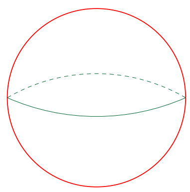
$$\int\int\int (2 + 2y + 2z)\, dV \tag{145}$$

Using spherical coordinates this is

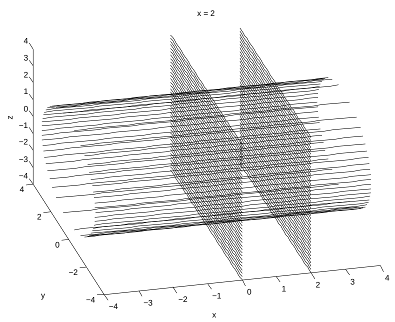
$$2\int_0^{2\pi}\int_0^{\pi}\int_0^{1} (1 + r\sin\theta\sin\varphi + r\cos\theta)\, r^2 \sin\theta dr d\theta d\varphi \tag{146}$$

The second and last terms vanish after integrating $\sin\varphi$ and $\cos\theta\sin\theta$ respectively. Therefore the answer is

$$2\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2 \sin\theta dr d\theta d\varphi = \frac{8\pi}{3} \tag{147}$$



Example 16. Verify the divergence theorem for $\mathbf{F} = 2x^2y\mathbf{i} - y^2\mathbf{j} + 4xz^2\mathbf{k}$ and the region bounded by $y^2 + z^2 = 9$, $x = 0$ and $x = 2$.



To compute the flux we use the divide the surface into the "lids", $x = 2$, $x = 0$ and the cylinder $y^2 + z^2 = 9$.

a) For $x = 2$, $\mathbf{F} = 8y\mathbf{i} - y^2\mathbf{j} + 8z^2\mathbf{k}$ and we can take $\mathbf{n} = \mathbf{i}$. Using polar coordinates

$$\begin{aligned}
&\int_0^{2\pi}\int_0^3 \left(8\rho\cos\varphi\mathbf{i} - \rho^2\cos^2\varphi\mathbf{j} + 8\rho^2\sin^2\varphi\mathbf{k}\right)\cdot\mathbf{i}\rho d\rho d\varphi \\
=&\int_0^{2\pi}\int_0^3 8\rho\cos\varphi\rho d\rho d\varphi \\
=&0
\end{aligned}$$

b) For $x = 0$, $\mathbf{F} = -y^2\mathbf{j}$ and we can take $\mathbf{n} = -\mathbf{i}$ (recall that the normal vector must point outwards). Using polar coordinates

$$\begin{aligned}
&\int_0^{2\pi}\int_0^3 \left(-\rho^2\cos^2\varphi\mathbf{j}\right)\cdot(-\mathbf{i})\rho d\rho d\varphi \\
=&\int_0^{2\pi}\int_0^3 8\rho\cos\varphi\rho d\rho d\varphi \\
=&0
\end{aligned}$$

c) Finally, for the cylinder $y^2 + z^2 = 9$ we take the normalized gradient as $\mathbf{n}$, that is,

$$\mathbf{n} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{y^2+z^2}} = \frac{y}{3}\mathbf{j} + \frac{z}{3}\mathbf{k} \tag{148}$$

Any point on the cylinder can be parameterized as $(x, y, z) = (x, 3\cos\varphi, 3\sin\varphi)$ so the surface differential becomes

$$|\mathbf{r}_x \times \mathbf{r}_\varphi| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & -3\sin\varphi & 3\cos\varphi \end{vmatrix} = |(0, -3\cos\varphi, -3\sin\varphi)| = 3 \tag{149}$$

$$dS = 3dxd\varphi \tag{150}$$

Hence the flux is

$$\begin{aligned}
&\int_0^{2\pi}\int_0^2 \left(6x^2\cos\varphi, -9\cos^2\varphi, 36x\sin^2\varphi\right)\cdot(0, \cos\varphi, \sin\varphi)\, 3dxd\varphi \\
=&3\int_0^{2\pi}\int_0^2 \left(-9\cos^3\varphi + 36x\sin^3\varphi\right) dxd\varphi \\
=&3\int_0^{2\pi} \left(-18\cos^3\varphi + 72\sin^3\varphi\right) d\varphi
\end{aligned}$$

Since sine and cosine are of period 2π then

$$3\int_0^{2\pi}\left(-18\cos^3\varphi + 72\sin^3\varphi\right) d\varphi = -54\int_{-\pi}^{\pi}\left(\cos^3\varphi - 4\sin^3\varphi\right) d\varphi \tag{151}$$

Using that sin is an odd function then $\sin^3\varphi$ is an odd function as well, so only the first term contributes

$$-54\int_{-\pi}^{\pi}\left(\cos^3\varphi-4\sin^3\varphi\right)d\varphi=-54\int_{-\pi}^{\pi}\cos^3\varphi d\varphi \tag{152}$$

Using the identity

$$\int\cos^3 axdx=\frac{3\sin ax}{4a}+\frac{\sin 3ax}{12a} \tag{153}$$

we find that this integral is zero as well and so the flux vanishes.

On the other hand, the divergence is

$$\nabla\cdot\mathbf{F}=4xy-2y+8xz \tag{154}$$

We use cylindrical coordinates (with the roles of the variables interchanged) so that $y=\rho\cos\varphi$, $z=\rho\sin\varphi$. We then need to integrate

$$\int_0^{2\pi}\int_0^3\int_0^2\left(4x\rho\cos\varphi-2\rho\cos\varphi+8x\rho\sin\varphi\right)\rho dxd\rho d\varphi \tag{155}$$

which you can check will vanish as well, verifying Gauss' divergence theorem.