# Integration of Scalar Fields

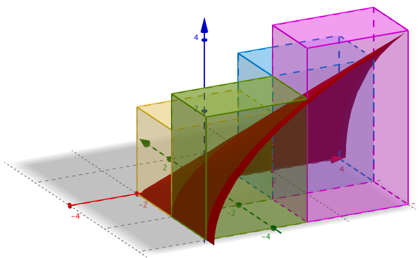
This material corresponds roughly to sections 15.1, 15.2, 15.3, 12.7, 15.4, 15.6 and 16.4 in the book.

**Double Integral:**

- Suppose that $z = f(x, y)$ is a function of two variables defined over a region $R$ of the $xy$ plane. We represent the double integral of $z$ over this region as

$$\int\int_R z(x, y)dA \tag{1}$$

Geometrically, we can think of it as a **net volume.** That is, when the surface $z(x, y)$ is above the $xy$ plane we consider the volume to be positive and when the surface $z(x, y)$ is below the $xy$ plane we consider the volume to be negative. Theoretically it is defined by approximating this integral by the volume of parallelepipeds with rectangular base and height given by the function as show in the figure.



**Fubini's Theorem:** suppose that $z(x,y)$ is continuous on the rectangle

$$R = [a,b] \times [c,d] = \{(x,y) \mid a \le x \le b, \quad c \le y \le d\} \tag{2}$$

Then the double integral $\int\int_R z(x,y)dA$ can be computed by integrating the **iterated integrals** $\int_a^b \left(\int_c^d z(x,y)dy\right) dx$ or $\int_c^d \left(\int_a^b z(x,y)dx\right) dy$, that is

$$\int\int_R z(x,y)dA = \int_a^b \left(\int_c^d z(x,y)dy\right) dx = \int_c^d \left(\int_a^b z(x,y)dx\right) dy \tag{3}$$

The second and third integrals are computed similarly to how one computes partial derivatives. For example, in $\int_a^b \left(\int_c^d z(x,y)dy\right) dx$ one integrates with respect to $y$ first, treating $x$ like a constant and then one integrates with respect to $x$. Likewise, for $\int_c^d \left(\int_a^b z(x,y)dx\right) dy$ one integrates with respect to $x$ first, treating $y$ like a constant.

For example, if $z(x,y) = \sqrt{2+x}\sqrt{-y}$ and $R = [-2,4] \times [-4,0]$ thanks to Fubini's Theorem

$$\int\int_R \sqrt{2+x}\sqrt{-y}dA = \int_{-2}^4 \left(\int_{-4}^0 \sqrt{2+x}\sqrt{-y}dy\right) dx = \int_{-4}^0 \left(\int_{-2}^4 \sqrt{2+x}\sqrt{-y}dx\right) dy \tag{4}$$

As mentioned above, $\int_{-2}^4 \left(\int_{-4}^0 \sqrt{2+x}\sqrt{-y}dy\right) dx$ means the following: integrate first with respect to $y$ pretending that $x$ is a constant and then integrate the result of the first integration with respect to $x$. If we follow the instructions then we can start by saying that

$$\int_{-2}^4 \left(\int_{-4}^0 \sqrt{2+x}\sqrt{-y}dy\right) dx = \int_{-2}^4 \sqrt{2+x} \left(\int_{-4}^0 \sqrt{-y}dy\right) dx \tag{5}$$

Because $\sqrt{2+x}$ is being treated as a constant with respect to the innermost integral and constants can be taken out of the integral. Since

$$\int_{-4}^0 \sqrt{-y}dy = -\frac{2}{3}(-y)^{3/2}\Big|_{y=-4}^{y=0} = \frac{2}{3}(4)^{3/2} = \frac{16}{3} \tag{6}$$

Then

$$\begin{aligned} \int_{-2}^4 \sqrt{2+x}\left(\int_{-4}^0 \sqrt{-y}dy\right) dx &= \int_{-2}^4 \sqrt{2+x}\left(\tfrac{16}{3}\right) dx \\ &= \tfrac{16}{3}\int_{-2}^4 \sqrt{2+x}dx \\ &= \tfrac{16}{3}\left(\tfrac{2}{3}(x+2)^{3/2}\right)\Big|_{x=-2}^{x=4} \\ &= \tfrac{32}{9}(6)^{3/2} \end{aligned} \tag{7}$$

Therefore $\int\int_R \sqrt{2+x}\sqrt{-y}dA = \frac{32}{9}(6)^{3/2}$. Notice that in this iterated integral we integrated first with respect to $y$. Thanks to Fubini's theorem we are free to compute the integration in the other order, that is, we can integrate first with respect to $x$.

The way in which we find $\int_{-4}^{0}\left(\int_{-2}^{4}\sqrt{2+x}\sqrt{-y}dx\right)dy$ is entirely analogous to the previous calculation. Now we are integrating first with respect to $x$ and so we treat $y$ as a constant. Therefore, we can say that

$$\int_{-4}^{0}\left(\int_{-2}^{4}\sqrt{2+x}\sqrt{-y}dx\right)dy=\int_{-4}^{0}\sqrt{-y}\left(\int_{-2}^{4}\sqrt{2+x}dx\right)dy \tag{8}$$

and now we have to find

$$\int_{-2}^{4}\sqrt{2+x}dx=\left(\frac{2}{3}(x+2)^{3/2}\right)\Big|_{x=-2}^{x=4}=\frac{2}{3}(6)^{3/2} \tag{9}$$

and in this way

$$\begin{aligned}\int_{-4}^{0}\sqrt{-y}\left(\int_{-2}^{4}\sqrt{2+x}dx\right)dy &= \int_{-4}^{0}\sqrt{-y}\left(\tfrac{2}{3}(6)^{3/2}\right)dy\\ &= \tfrac{2}{3}(6)^{3/2}\int_{-4}^{0}\sqrt{-y}dy\\ &= \tfrac{2}{3}(6)^{3/2}\left(\tfrac{16}{3}\right)\\ &= \tfrac{32}{9}(6)^{3/2}\end{aligned} \tag{10}$$

and we see that we obtain the same answer.

**Example 1.** Find $\int\int_R(4-x-y)dA$ where $R=[0,2]\times[0,1]$

Again we will compute both iterated integrals $\int_0^2\int_0^1(4-x-y)dydx$ and $\int_0^1\int_0^2(4-x-y)dxdy$ to show that both integrals agree. In practice, however, it is only necessary to compute only of the iterated integrals.

$$\begin{aligned}\int_0^2\int_0^1(4-x-y)dydx &= \int_0^2\left(4y-xy-\tfrac{y^2}{2}\right)\Big|_{y=0}^{y=1}dx\\ &= \int_0^2\left(\tfrac{7}{2}-x\right)dx\\ &= \left(\tfrac{7}{2}x-\tfrac{x^2}{2}\right)\Big|_{x=0}^{x=2}\\ &= 5\end{aligned} \tag{11}$$

The second iterated integral is

$$\begin{aligned}\int_0^1\int_0^2(4-x-y)dxdy &= \int_0^1\left(4x-\tfrac{x^2}{2}-yx\right)\Big|_{x=0}^{x=2}dy\\ &= \int_0^1(6-2y)\,dy\\ &= \int_0^1\left(6y-y^2\right)\Big|_{y=0}^{1}\\ &= 5\end{aligned} \tag{12}$$

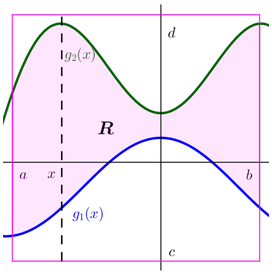
**Integration over non-rectangular regions**:

- **Type I Region**: suppose $g_1(x)$ and $g_2(x)$ are continuous functions on $[a,b]$ and the region $R$ is defined by

$$R = \{(x,y) \mid g_1(x) \le y \le g_2(x), a \le x \le b\} \tag{13}$$

Then

$$\int\int_R f(x,y)dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right] dx \tag{14}$$

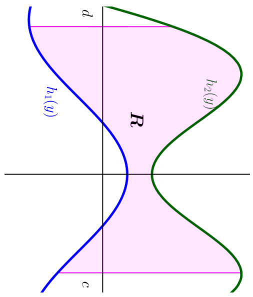



- **Type II Region**: suppose $h_1(y)$ and $h_2(y)$ are continuous functions on $[c,d]$ and the region $R$ is defined by

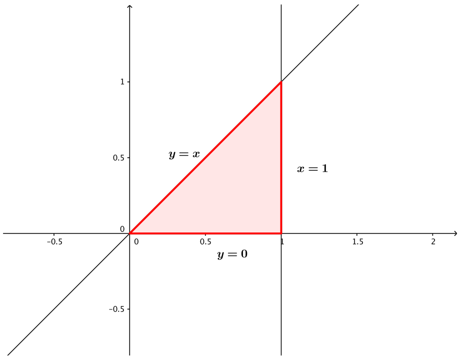
$$R = \{(x,y) \mid h_1(y) \le x \le h_2(y); c \le y \le d\} \tag{15}$$

Then

$$\int\int_R f(x,y)dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)dx\right] dy \tag{16}$$




For example, suppose we want to find $\int\int_R \frac{\sin x}{x} dA$ were $R$ is triangle whose sides lie down on the $x$ axis and the lines $y = x$ , $x = 1$.




Observe that this region is simultaneously type I and type II. We start by treating it as a type I region. In this case $x$ takes its values on the interval $[0, 1]$. If we fix an $x$ inside this interval, then the values that $y$ takes start at the horizontal line $y = 0$ and end at the straight line $y = x$. Therefore the integral becomes

$$\begin{aligned}
\int\int_R \tfrac{\sin x}{x} dA &= \int_0^1 \int_0^x \tfrac{\sin x}{x} dy dx \\
&= \int_0^1 \left(\tfrac{\sin x}{x}\right) \int_0^x dy dx \\
&= \int_0^1 \left(\tfrac{\sin x}{x}\right) x dx \\
&= \int_0^1 \sin x dx \\
&= -\cos x|_{x=0}^{x=1} \\
&= 1 - \cos 1
\end{aligned} \tag{17}$$

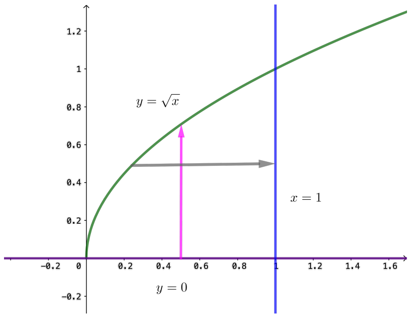
If we treat the region as type II then we use that $y$ takes its values on the interval $[0, 1]$. For a fixed $y$, the values $x$ take start at $x = y$ and end at $x = 1$ and so the integral can be written as

$$\int\int_R \frac{\sin x}{x} dA = \int_0^1 \int_y^1 \frac{\sin x}{x} dx dy \tag{18}$$

Notice that in this case we would need to find an antiderivative of $\frac{\sin x}{x}$ which is no easy task and so from a practical point of view this integral can't be computed. This example shows that despite the fact that the region is both of type I and type II, it may not be equally easy to find the corresponding iterated integrals.

---

**Example 2. Find $\int\int_R xy dA$ , where $R$ is the region bounded by the curves $y = 0$, $x = 1$, and $y = \sqrt{x}$.**




If we use the order of integration $dA = dydx$, then we can have to think of the region of integration as being composed of vertical segments (the pink arrow in the figure).

Notice that these arrows all start on the $x$-axis (equation $y = 0$), and they all end on the green curve (equation $y = \sqrt{x}$). Therefore the integral we must do is

$$\int\int_R xydA$$
$$= \int_?^? \left( \int_0^{\sqrt{x}} xydy \right) dx$$

We still need to determine the bounds for the exterior integral. These correspond to smallest and largest value $x$ takes in this region. Therefore, since $0 \leq x \leq 1$ we have to do the double integral

$$\int\int_R xydA$$
$$= \int_0^1 \left( \int_0^{\sqrt{x}} xydy \right) dx$$
$$= \int_0^1 \left( x\frac{y^2}{2} |_{y=0}^{y=\sqrt{x}} \right) dx$$
$$= \frac{1}{2} \int_0^1 x(x)dx$$
$$= \frac{1}{2} \left( \frac{x^3}{3} \right) |_{x=0}^{x=1}$$
$$= \frac{1}{6}$$

On the other hand, we can also use the order of integration $dA = dxdy$. In this case, we think of the region of integration as being made of horizontal segments. They all begin on the green curve (equation $y = \sqrt{x}$) and end on the blue curve (equation $x = 1$). Notice that when we use the order $dxdy$, $x$ is being regarded as a function of $y$, which means that the equation of the green curve is $x = y^2$ instead of $y = \sqrt{x}$.

In this case we have that

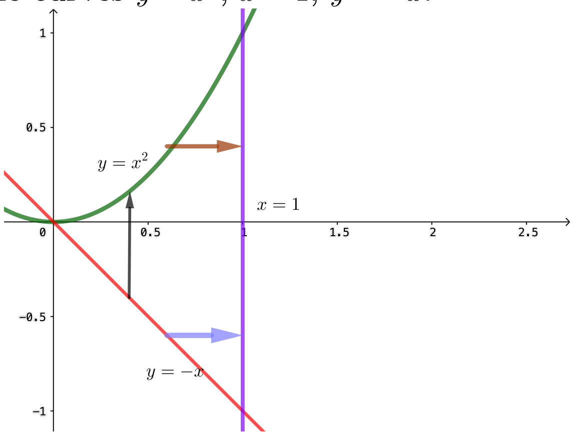
$$\begin{aligned}&\int\int_R xydA\\ =&\int_?^? \left(\int_{y^2}^1 xydx\right) dy\end{aligned}$$

The bounds for $y$ are the smallest and largest value $y$ takes in the region of integration. Since $0 \leq y \leq 1$ the integral we must do is

$$\begin{aligned}&\int\int_R xydA\\ =&\int_0^1 \left(\int_{y^2}^1 xydx\right) dy\\ =&\int_0^1 y\left(\frac{x^2}{2}\right) |_{x=y^2}^{x=1}\ dy\\ =&\frac{1}{2}\int_0^1 y(1-y^4)dy\\ =&\frac{1}{2}\int_0^1 (y-y^5)dy\\ =&\frac{1}{2}(\frac{y^2}{2}-\frac{y^6}{6})\ |_{y=0}^{y=1}\\ =&\frac{1}{2}\left(\frac{3-2}{6}\right)\\ =&\frac{1}{6}\end{aligned}$$

---

**Example 3. Find the double integral $\int\int_R y^2xdA$ where $R$ is the region between the curves $y=x^2$, $x=1$, $y=-x$.**




If we use the order of integration $dA = dydx$ then we use vertical arrows to describe the region of integration, which means that

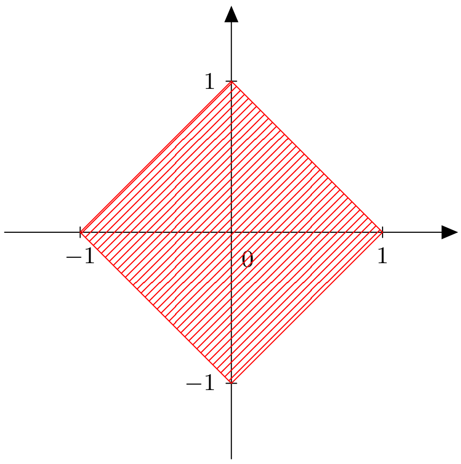
$$\begin{aligned}
&\int\int y^2 x dA \\
=&\int_0^1 \left( \int_{-x}^{x^2} y^2 x dy \right) dx \\
=&\int_0^1 x \left( \frac{y^3}{3} \right) |_{y=-x}^{y=x^2} dx \\
=&\frac{1}{3} \int_0^1 x \left( x^6 + x^3 \right) dx \\
=&\frac{1}{3} \int_0^1 (x^7 + x^4) dx \\
=&\frac{1}{3} \left( \frac{x^8}{8} + \frac{x^5}{5} \right) |_0^1 \\
=&\frac{13}{120}
\end{aligned}$$

On the other hand, if we use the order of integration $dA = dxdy$, then we need to split the double integral into two double integrals, since the horizontal segments can start from two different curves: the green and red curves. Therefore

$$\begin{aligned}
&\int\int y^2 x dA \\
=&\int_0^1 \left(\int_{\sqrt{y}}^1 y^2 x dx\right) dy + \int_{-1}^0 \left(\int_{-y}^1 y^2 x dx\right) dy \\
=&\int_0^1 \left(\int_{\sqrt{y}}^1 y^2 x dx\right) dy + \int_{-1}^0 \left(\int_{-y}^1 y^2 x dx\right) dy \\
=&\int_0^1 y^2 \frac{x^2}{2} \mid_{x=\sqrt{y}}^{x=1} dy + \int_{-1}^0 y^2 \frac{x^2}{2} \mid_{x=-y}^{x=1} dy \\
=&\frac{1}{2}\int_0^1 y^2(1-y)dy + \frac{1}{2}\int_{-1}^0 y^2(1-y^2)dy \\
=&\frac{1}{2}\int_0^1 (y^2-y^3)dy + \frac{1}{2}\int_{-1}^0 (y^2-y^4)dy \\
=&\frac{1}{2}\left(\frac{y^3}{3}-\frac{y^4}{4}\right) \mid_{y=0}^{y=1} + \frac{1}{2}\left(\frac{y^3}{3}-\frac{y^5}{5}\right) \mid_{y=-1}^{y=0} \\
=&\frac{1}{2}\left(\frac{1}{3}-\frac{1}{4}\right) + \frac{1}{2}\left(0-\left(-\frac{1}{3}+\frac{1}{5}\right)\right) \\
=&\frac{1}{2}\left(\frac{1}{3}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}\right) \\
=&\frac{40-15-12}{120} \\
=&\frac{13}{120}
\end{aligned}$$

---

**Example 4.** Find $\int\int_R \frac{x+y}{x+y+2} dA$ where $R$ is the region shown in the following figure.




We will treat this region as a type I region. In this case $x$ takes values on the interval $[-1, 1]$. For a fixed $x$, $y$ takes values between either the lower left - upper left sides of the rhombus or the lower right - upper right sides of the rhombus. This depends on whether $x$ is positive or negative. Therefore, the first thing to do is find the equations of the four lines. These are

$$\begin{array}{cc} \text{lower left} & y=-x-1 \\ \text{upper left} & y=x+1 \\ \text{lower right} & y=x-1 \\ \text{upper right} & y=-x+1 \end{array} \tag{19}$$

Therefore, the integral is equal to

$$\int\int_R \frac{x+y}{x+y+2}dA = \int_{-1}^{0}\int_{-x-1}^{x+1}\frac{x+y}{x+y+2}dydx + \int_0^1\int_{x-1}^{-x+1}\frac{x+y}{x+y+2}dydx \tag{20}$$

To find the integral we make the substitution (remember that $x$ is constant)

$$\begin{array}{c} u=x+y+2 \\ du=dy \end{array} \tag{21}$$

and so the integrals become

$$\begin{array}{rc} & \int_{-1}^{0}\int_{1}^{2x+3}\frac{u-2}{u}dudx + \int_0^1\int_{2x+1}^{3}\frac{u-2}{u}dudx \\ = & \int_{-1}^{0}\int_{1}^{2x+3}\left(1-\frac{2}{u}\right)dudx + \int_0^1\int_{2x+1}^{3}\left(1-\frac{2}{u}\right)dudx \\ = & \int_{-1}^{0}\left(2x+2-2\ln(2x+3)\right)dx \\ & +\int_0^1(2-2x-2\left(\ln 3-\ln(2x+1)\right)dx \end{array} \tag{22}$$

Using an substitution and integration by parts we can show that

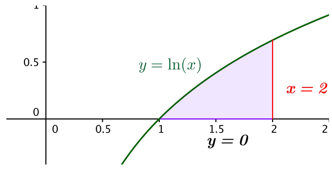
$$\begin{array}{c} \int \ln(2x+3)dx = \frac{1}{2}(2x+3)(-1+\ln(2x+3))+C \\ \int \ln(2x+1)dx = \frac{1}{2}(2x+1)(-1+\ln(2x+1))+C \end{array} \tag{23}$$

and so the last integrals equal

$$\begin{array}{rc} & \left(x^2+2x-(2x+3)(-1+\ln(2x+3))\right)\Big|_{x=-1}^{x=0} \\ & +\left(2x-x^2-2x\ln 3+(2x+1)(-1+\ln(2x+1))\right)\Big|_{x=0}^{x=1} \\ = & -3(-1+\ln 3)-[1-2-(1)(-1+\ln 1)] \\ & +1-2\ln 3+3(-1+\ln 3)-1(-1+\ln 1) \\ = & 3-3\ln 3 \\ & +2-2\ln 3-3+3\ln 3 \\ = & 2-2\ln 3 \end{array} \tag{24}$$

---

**Problem 5.** Write the double integral $\int_1^2\int_0^{\ln x}(x-1)\sqrt{1+e^{2y}}dydx$ in the order $dxdy$. Do not find the value.



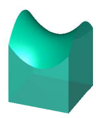

The way the integral is written, we need to find an antiderivative of $\sqrt{1+e^{2y}}$, which is not impossible but it is somewhat tedious. Therefore, we will invert the order of integration.

To do this notice that the largest value of $y$ in the region of integration is $\ln 2$ and so the interval of integration for $y$ is $[0, \ln 2]$. For a fixed value of $y$, $x$ starts at the curve $y = \ln x$, which we write as $x = e^y$, and ends at the curve $x = 2$. Therefore the integral is the same as

$$\int_0^{\ln 2} \int_{e^y}^{2} (x-1)\sqrt{1+e^{2y}}dxdy \tag{25}$$

Let $R$ be a region in the $xy$ plane and let $f$ be continuous and nonnegative on $R$. Then the **volume** of the solid under a surface bounded above by $z = f(x, y)$ and below by $R$ is given by

$$V = \int_R \int f(x, y)dA \tag{26}$$



**Problem 6.** Find the volume of the solid that lies under $z = x^2 + y^2$ and above the square $0 \le x \le 2$, $-1 \le y \le 1$.

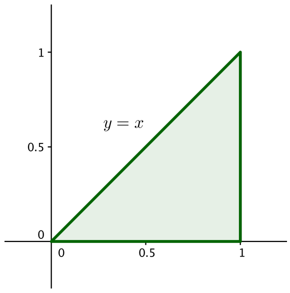
The volume is

$$\begin{aligned} V &= \int_0^2 \int_{-1}^1 (x^2+y^2)dydx \\ &= \int_0^2 \left(x^2y + \frac{y^3}{3}\right)\Big|_{y=-1}^{y=1} dx \\ &= \int_0^2 \left(2x^2 + \frac{2}{3}\right) dx \\ &= \left(\frac{2x^3}{3} + \frac{2}{3}x\right)\Big|_{x=0}^{x=2} \\ &= \frac{20}{3} \end{aligned} \tag{27}$$

f $f$ is integrable over the plane region $R$, then its average value over $R$ is given by

$$\frac{\int_R \int f(x,y)dA}{\text{area of } R} = \frac{\int_R \int f(x,y)dA}{\int_R \int dA} \tag{28}$$

**Problem 7.** Find the average value of the function $f(x,y) = e^{-x^2}$ over the plane region $R$: the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.




The area of the triangle is $\frac{1}{2}$ and so the average value is

$$\begin{aligned} & \frac{\int_0^1 \int_0^x e^{-x^2} dydx}{\frac{1}{2}} \\ = \quad & 2\int_0^1 xe^{-x^2} dx \\ = \quad & \left(-e^{-x^2}\right)\Big|_{x=0}^{x=1} \\ = \quad & (1-e^{-1}) \end{aligned} \tag{29}$$

When integrating a scalar function (like density) $\lambda(s)$ along a curve $\mathbf{r}(s)$ , both parameterized by the arc length, we need to compute

$$m = \int_0^L \lambda ds \tag{30}$$

If we use a parameterization which is not $s$, then we compute instead

$$m = \int_a^b \lambda v dt \tag{31}$$

where $[a,b]$ is the interval which parameterizes the curve.

**Example 8.** Suppose a cable has circular shape and it has radius $R$, centered at the origin. If the density of the cable is $\lambda(x,y) = x^4 + x^2y^2$, find the total mass.

We will use the second formula, for which we will use the parameterization

$$\mathbf{r}(t) = R\cos t\mathbf{i} + R\sin t\mathbf{j} \tag{32}$$

of the circle, for $0 \le t < 2\pi$. In this case $v = \left|\mathbf{v}\right| = R$. Since

$$x = R\cos t \quad y = R\sin t \tag{33}$$

the density as a function of $t$ is

$$\lambda = R^4\cos^4 t + R^4\cos^2 t\sin^2 t = R^4\cos^2 t \tag{34}$$

Using equation 31 we find that

$$m = \int_0^{2\pi} R^5\cos^2 t dt = R^5\pi \tag{35}$$

---

**Polar Coordinates: "official" convention**

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{y}{x}$$

$$\boxed{dA = rdrd\theta} \tag{36}$$

**Polar Coordinates: Other acceptable conventions [picture]**

$$x = r\cos\varphi \qquad y = r\sin\varphi$$
$$r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{y}{x}$$

$$\boxed{dA = rdrd\varphi} \tag{37}$$

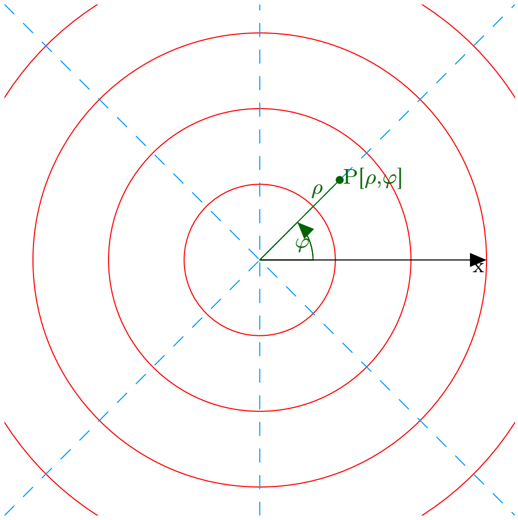



Figure 1: Polar coordinates $\rho, \varphi$ on the $xy$ plane

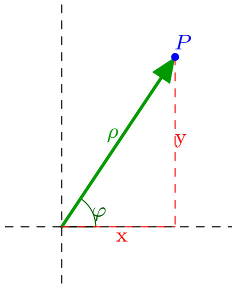



Figure 2: Polar and cartesian coordinates

**Cylindrical Coordinates: "official" convention**

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$
$$r = \sqrt{x^2+y^2} \qquad \tan\varphi = \frac{y}{x}$$

$$\boxed{dV = rdzdrd\theta} \tag{38}$$

**Cylindrical Coordinates: other acceptable conventions [picture]**

$$x = r\cos\varphi \qquad y = r\sin\varphi \qquad z = z$$
$$r = \sqrt{x^2+y^2} \qquad \tan\varphi = \frac{y}{x}$$

$$\boxed{dV = rdzdrd\varphi} \tag{39}$$

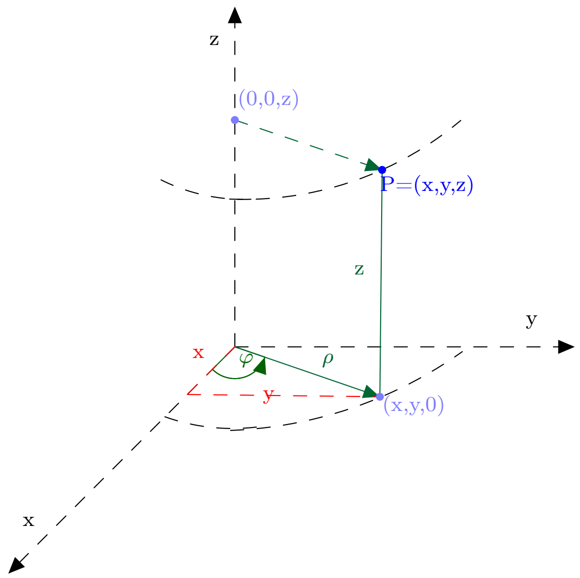



Figure 3: Cartesian and cylindrical coordinates

**Spherical Coordinates: official conventions**

$$x = \rho\cos\theta\sin\varphi \qquad y = \rho\sin\theta\sin\varphi \qquad z = \rho\cos\varphi$$

$$\rho = \sqrt{x^2+y^2+z^2} \qquad \tan\theta = \frac{y}{x} \qquad \cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Here $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$.

$$\boxed{dV = \rho^2 \sin\varphi d\rho d\varphi d\theta} \tag{40}$$

**Spherical Coordinates: other acceptable conventions [picture]**

$$x = r\cos\varphi\sin\theta \qquad y = r\sin\varphi\sin\theta \qquad z = r\cos\theta$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \tan\varphi = \frac{y}{x} \qquad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Here $0 \le \varphi \le 2\pi$ and $0 \le \theta \le \pi$.

$$\boxed{dV = r^2 \sin\theta dr d\theta d\varphi} \tag{41}$$

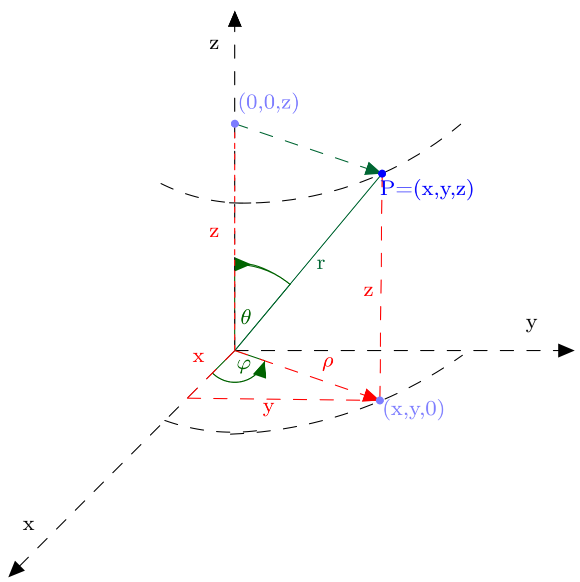



Figure 4: Spherical Coordinates

**Example 9. Find the triple integral**

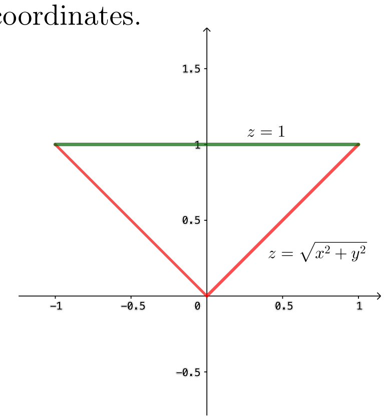
$$\int\int\int_R zdV \tag{42}$$

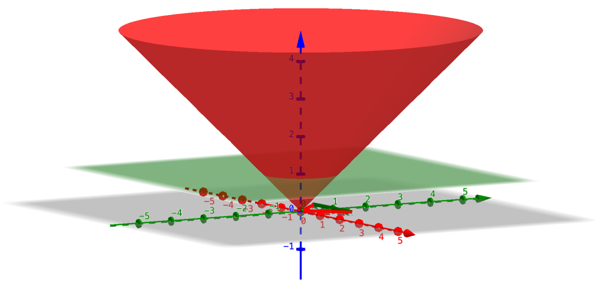
**where $R$ is the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$.**

**Do the integral using cylindrical and spherical coordinates**.

The region of integration is between the $xy$ plane and the $z = 1$ plane, as shown on right hand side image. Since the region of integration is symmetric about the $z$ axis, one can take a cross section of the region, obtained by intersecting the region with *any* plane which is perpendicular to the $xy$ plane and contains the $z$ axis: this is the left hand side image.

Notice that the vertical axis corresponds to the $z$ axis, while the horizontal axis can be regarded as representing the $r$ axis, where $r$ denotes the radial coordinate from cylindrical coordinates.



**Cylindrical coordinates:** here finding the bounds for the order of integration $dzdr$ can be obtained in the same way in which we found the bounds for the order of integration $dydx$ when working on a non-rectangular region ($z$ plays the role of $y$, while $r$ plays the role of $x$). In cylindrical coordinates $z = \sqrt{x^2 + y^2} = r$ so

$$\begin{aligned}
&\int\int\int_R zdV \\
=&\int\int\int_R zrdzdrd\theta \\
=&\int_0^{2\pi}\left(\int_0^1\left(\int_r^1 zrdz\right)dr\right)d\theta \\
=&\int_0^{2\pi}\int_0^1 r\frac{z^2}{2}\,|_{z=r}^{z=1}\,drd\theta \\
=&\frac{1}{2}\int_0^{2\pi}\int_0^1 (r - r^3)drd\theta \\
=&\frac{1}{2}\int_0^{2\pi}(\frac{r^2}{2} - \frac{r^4}{4})\,|_{r=0}^{r=1}\,d\theta \\
=&\frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)\int_0^{2\pi} d\theta \\
=&\frac{\pi}{4}
\end{aligned}$$

On the other hand, when using spherical coordinates we have [notice that the equation $z = \sqrt{x^2 + y^2}$ reads in spherical coordinates $\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} = \rho\sin\phi$ from which we obtain $\phi = \pi/4$ as one of the bounds for $\phi$. Likewise, $z = 1$ reads

$\rho \cos\phi = 1$ or $\rho = \frac{1}{\cos\phi}$, which is one of the other bounds we cared about].

$$
\begin{aligned}
&\int\int\int_R z dV \\
=&\int\int\int_R z\rho^2 \sin\phi d\rho d\phi d\theta \\
=&\int_0^{2\pi}\left(\int_0^{\pi/4}\left(\int_0^{\frac{1}{\cos\phi}}(\rho\cos\phi)\rho^2\sin\phi d\rho\right)d\phi\right)d\theta \\
=&2\pi\int_0^{\pi/4}\frac{\rho^4}{4}\,|_{\rho=0}^{\rho=\frac{1}{\cos\phi}}\ \cos\phi\sin\phi d\phi \\
=&\frac{\pi}{2}\int_0^{\pi/4}\frac{\cos\phi\sin\phi}{\cos^4\phi}d\phi \\
=&\frac{\pi}{2}\int_0^{\pi/4}\frac{\sin\phi}{\cos^3\phi}d\phi \\
=&\frac{\pi}{2}\left(\frac{1}{2}\right)(\cos^{-2}\phi)\,|_{\phi=0}^{\phi=\pi/4} \\
=&\frac{\pi}{4}\left(\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}-\frac{1}{1^2}\right) \\
=&\frac{\pi}{4}
\end{aligned}
$$

Notice that to find $\int_0^{\pi/4}\frac{\sin\phi}{\cos^3\phi}d\phi$ you can make a substitution $u = \cos\phi$, $du = -\sin\phi d\phi$.

**Computing surface integrals:**
If a surface $S$ is parameterized with coordinates $u, v$ then the surface differential is

$$dS = \left|\frac{\partial \mathbf{r}(u,v)}{\partial u} \times \frac{\partial \mathbf{r}(u,v)}{\partial v}\right| dudv \tag{43}$$

and the surface integral of scalar field $f$ is denoted as

$$\int\int_S f(u,v)dS \tag{44}$$

When $f = f(x,y)$ is a function defined on a region $R$ inside the $xy$ plane we can write the previous integral as

$$\int\int_R f(x,y)dxdy \tag{45}$$

The area of the surface can be computed as

$$A(S) = \int\int_S dS \tag{46}$$

When we are parameterizing the region $R$ using coordinates $u, v$ then we write $dS$ as

$$dA = J(u,v)dudv \tag{47}$$

where $J(u,v)$ is the **Jacobian** of the parameterization (or of the change of coordinates)

$$J(u,v) \equiv \frac{\partial(x,y)}{\partial(u,v)} \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \tag{48}$$

In this way the integral can be computed as

$$\int\int f(u,v)J(u,v)dudv \tag{49}$$

As an example, if we use polar coordinates then the area differential can be written as
El diferencial de área en coordenadas polares es

$$dA = rdrd\theta \tag{50}$$

and the integral to perform can be written as

$$\int\int frdrd\theta \tag{51}$$

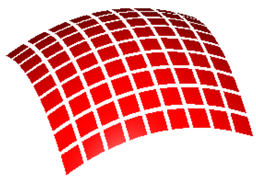


Figure 5: Surface Integral

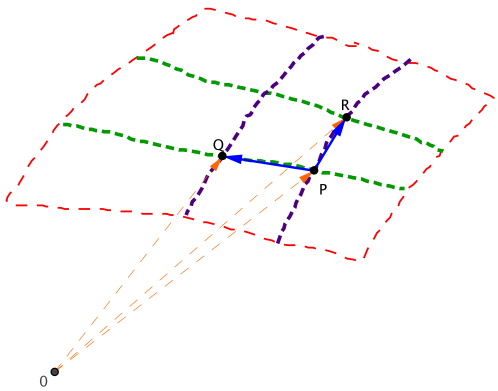



Figure 6: Computing a surface integral

**Example 10. Evaluate** $\int_0^a \int_0^{\sqrt{ax-x^2}} \frac{a dy dx}{\sqrt{a^2-x^2-y^2}}$

The idea is to compute this using polar coordinates since it simplifies the integrand, as well as the bounds of integration. Notice that

$$0 \leq x \leq a \qquad 0 \leq y \leq \sqrt{ax - x^2} \tag{52}$$

To interpret the second inequality we square it first

$$0 \leq y^2 \leq ax - x^2 \tag{53}$$

adding $x^2$ to each side of the inequality we conclude that

$$x^2 \leq x^2 + y^2 \leq ax \tag{54}$$

The first inequality says that

$$x^2 = x^2 + y^2 \tag{55}$$

which is equivalent to $y = 0$.

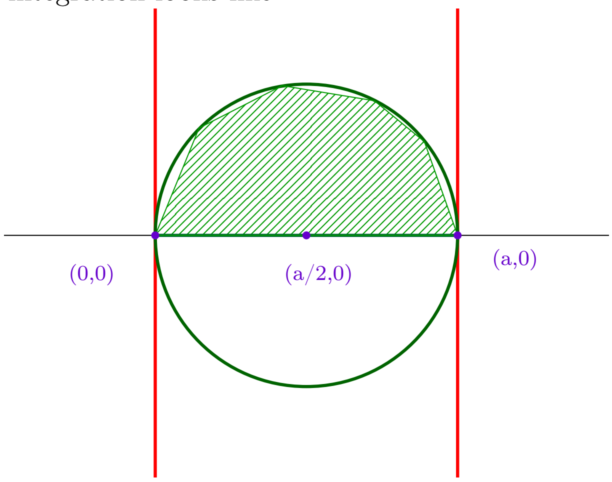
The second inequality says that

$$x^2 + y^2 = ax \tag{56}$$

Completing squares we find that

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} \tag{57}$$

which is the equation of a circle centered at $\left(\frac{a}{2}, 0\right)$ with radius $\frac{a}{2}$. Therefore, the region of integration looks like




Now we need to find the limits with respect to the variables $r, \theta$. From the picture

$$0 \leq \theta \leq \frac{\pi}{2} \tag{58}$$

Using $x^2 + y^2 \leq ax$ we can write this as

$$r \leq a \cos\theta \tag{59}$$

so the integral becomes

$$\int_0^a \int_0^{\sqrt{ax-x^2}} \frac{a}{\sqrt{a^2-x^2-y^2}} dA = \int_0^{\frac{\pi}{2}} \left( \int_0^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}} r dr \right) dr \tag{60}$$

Using the change of variables

$$u = a^2 - r^2 \qquad du = -2r dr \tag{61}$$

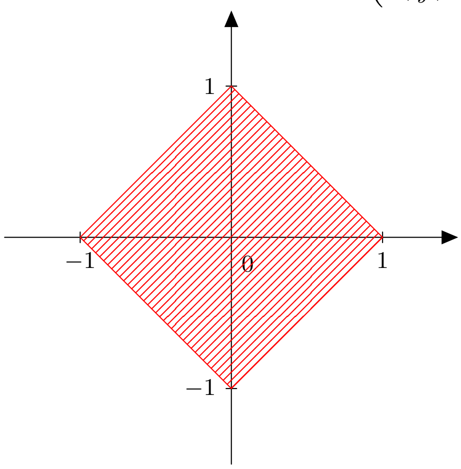
we find that

$$\int_0^{\frac{\pi}{2}} \left( \int_{a^2}^{a^2\sin^2\theta} \frac{a}{\sqrt{u}} \left( -\frac{du}{2} \right) \right) d\theta = a \int_0^{\frac{\pi}{2}} \sqrt{u}\Big|_{a^2\sin^2\theta}^{a^2} d\theta \tag{62}$$

$$= a \int_0^{\frac{\pi}{2}} (a - a\sin\theta)\, d\theta = a^2 (\theta + \cos\theta)\Big|_0^{\frac{\pi}{2}} = a^2 \left( \frac{\pi}{2} - 1 \right) \tag{63}$$

---

**Example 11. Evaluate $\int\int_R \left(\frac{x-y}{x+y+2}\right)^2 dxdy$ where $R$ is the region shown below.**




The region is bounded by the four lines

$$\begin{array}{cc} x+y=1 & x-y=1 \\ x+y=-1 & x-y=-1 \end{array} \tag{64}$$

This suggests making the change of variables

$$u = x + y \quad v = x - y \tag{65}$$

which we rewrite as

$$x = \frac{u+v}{2} \quad y = \frac{u-v}{2} \tag{66}$$

The Jacobian is

$$J(u,v) = \left|\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \left|\det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right| = \left|-\frac{1}{2}\right| = \frac{1}{2} \tag{67}$$

Recall that we take the absolute values to make it positive. Finally

$$\begin{aligned}
&\int\int_R \left(\frac{x-y}{x+y+2}\right)^2 dA \\
=&\int_{-1}^{1}\left(\int_{-1}^{1}\left(\frac{v}{u+2}\right)^2 \frac{1}{2}du\right)dv \\
=&\int_{-1}^{1} -\frac{1}{2}v^2\left(u+2\right)^{-1}\Big|_{-1}^{1} dv \\
=&-\frac{1}{2}\left(\frac{1}{3}-1\right)\int_{-1}^{1} v^2 dv \\
=&\frac{1}{3}\frac{v^3}{3}\Big|_{-1}^{1} \\
=&\frac{2}{9}
\end{aligned}$$

---

**Computation of volume:**

If a region $V$ is parameterized using coordinates $u, v, w$ then the volume differential is

$$dV = \left|\frac{\partial \mathbf{r}(u,v,w)}{\partial u}\cdot\left(\frac{\partial \mathbf{r}(u,v,w)}{\partial v}\times\frac{\partial \mathbf{r}(u,v,w)}{\partial w}\right)\right|dudvdw \tag{68}$$

If $\mathbf{r} = x(u,v,w)\mathbf{i}+y(u,v,w)\mathbf{j}+z(u,v,w)\mathbf{k}$ this volume differential can also be computed as

$$dV = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{pmatrix}\right| dudvdw \equiv |J(u,v,w)|dudvdw \tag{69}$$

where $J(u,v,w)$ is the Jacobian in three variables. We denote the integral as

$$\int\int\int_V f(u,v,w)dV \tag{70}$$

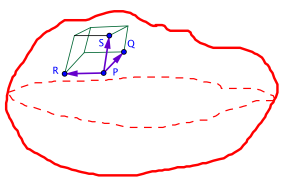



Figure 7: Computing volume

When the region of integration is defined by the conditions

$$a \leq x \leq b \quad g_1(x) \leq y \leq g_2(x) \quad h_1(x,y) \leq z \leq h_2(x,y) \tag{71}$$

the integral is written as

$$\int_a^b \left( \int_{g_1(x)}^{g_2(x)} \left( \int_{h_1(x,y)}^{h_2(x,y)} F(x,y,z)dz \right) dy \right) dx \tag{72}$$

with analogous formulas if the roles of $x, y, z$ are changed.

**Example 12.** Find the volume of solid $T$ limited by the paraboloids $z = x^2 + y^2$, $z = 4x^2 + 4y^2$, the cylinder $y = x^2$ , and the plane $y = 3x$.

For convenience we break the picture into two, the first one denotes the “floor” and the “roof” for $T$, while the second denotes the “lateral walls”.

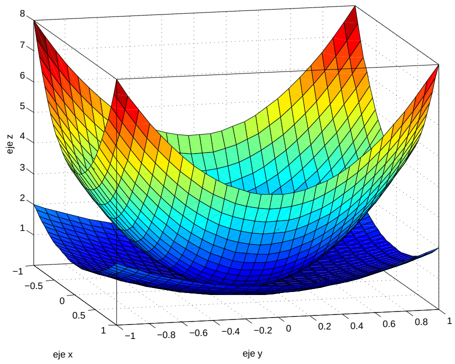



Figure 8: Volume between paraboloids

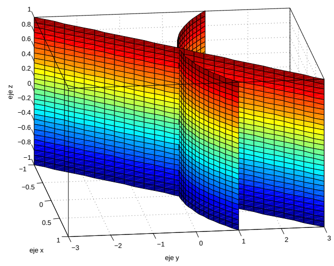



Figure 9: Region determined by the cylinder and the plane

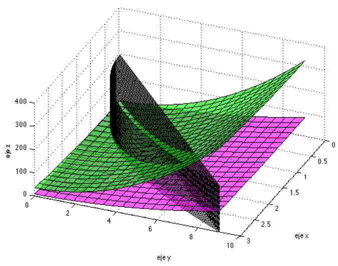



Figure 10: Picture depicting the four surfaces at once

To find the limits of integration we also draw things on the $xy$ plane.

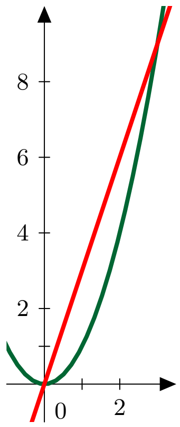



Figure 11: Region of integration on the $xy$ plane

The curves $y = x^2$ and $y = 3x$ intersect when $x^2 = 3x$, that is, $x = 0, x = 3$. Using formula 72 we find that the volume is

$$
\begin{aligned}
& V \\
= & \int\int\int dzdydx \\
= & \int_0^3 \left( \int_{x^2}^{3x} \left( \int_{x^2+y^2}^{4(x^2+y^2)} dz \right) dy \right) dx \\
= & \int_0^3 \left( \int_{x^2}^{3x} 3(x^2+y^2) dy \right) dx \\
= & \int_0^3 \left(3x^2y + y^3\right) |_{x^2}^{3x} dx \\
= & \int_0^3 3x^2\left(3x - x^2\right) + 27x^3 - x^6 dx \\
= & \int_0^3 36x^3 - 3x^4 - x^6 dx \\
= & \frac{16767}{35}
\end{aligned}
$$

---

**Example 13. Find the volume of a sphere of radius $R$.**
In this case we use spherical coordinates

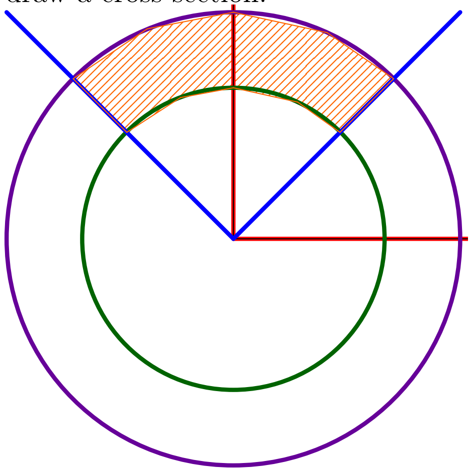
$$V = \int\int\int dV = \int_0^{2\pi}\int_0^{\pi}\int_0^R r^2 \sin\theta dr d\theta d\varphi = \frac{4}{3}\pi R^3 \tag{73}$$

---

**Example 14.** Use spherical coordinates to evaluate $\int\int\int_T \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}}$, where $T$ is the region bounded by the spheres $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 = 9$ and the half-cone

$x^2 + y^2 - z^2 = 0$, $z \geq 0$.

With respect to spherical coordinates the previous equations become $r = 2$, $r = 3$ and $\sin^2\theta = \cos^2\theta$. These equations are independent of $\varphi$, so our region can be regarded as being obtained from a slice which is rotated with respect to the $z$ axis, so we just need to draw a cross section.



From here it is clear that the limits are

$$2 \leq r \leq 3 \qquad 0 \leq \theta \leq \frac{\pi}{4} \qquad 0 \leq \varphi \leq 2\pi \tag{74}$$

so the integral in spherical coordinates is

$$\begin{aligned}
&\int_0^{2\pi} \left( \int_0^{\frac{\pi}{4}} \left( \int_2^3 \frac{r^2 \sin\theta}{r^3} dr \right) d\theta \right) d\varphi \\
=&2\pi \left(\ln r\right) |_2^3 \left(-\cos\theta\right) |_0^{\frac{\pi}{4}} \\
=&2\pi \left(1 - \frac{1}{\sqrt{2}}\right) \ln\left(\frac{3}{2}\right)
\end{aligned}$$