Additional Problems for Multivariable Calculus

Mariano Echeverria

1 Vectors and Curves

1. Two lines on the xy plane either intersect at a point or are parallel. a) When you have two lines in 3d space (so xyz space), what are the possibilities? b) What about two planes in 3d space? c) What about three planes in 3d space? d) Or four planes in 3d space? e) What about a line and a plane in 3d space?

2. Two lines on the xy plane determine a unique line passing through these points. a) Is this still true in 3d space? b) How many points do you need to specify a plane in 3d space?

3. Suppose $\mathbf{v}$ is a vector with certain physical units. For example, if $\mathbf{v}$ were a velocity vector then its units would be length/time. How are the units of $|\mathbf{v}|$ related to the units of $\mathbf{v}$? What are the units of $\frac{\mathbf{v}}{|\mathbf{v}|}$?

4. What is the line of intersection between the xz and yz planes?

5. What does the equation $x = 3$ correspond to if a) x is the only variable being considered, b) x, y are the only variables being considered, c) x, y, z are the only variables being considered.

6. What does the equation $x^2 + y^2 = 4$ correspond to if a) x, y are the only variables being considered, b) x, y, z are the only variables being considered.

7. Suppose $\mathbf{v}$ is a vector on the xy plane different from $\mathbf{0}$. Then $\mathbf{v} = |\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$ is known as the *polar decomposition* of the vector. Can you think of a reason why this name is given to the previous equation? Hint: think of complex numbers.

8. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$ are two vectors and $\mathbf{u}$ another vector such that $\text{proj}_{\mathbf{u}}\mathbf{v}_1 = 3\mathbf{i} - 5\mathbf{k}$ and $\text{proj}_{\mathbf{u}}\mathbf{v}_2 = 2\mathbf{j} + 7\mathbf{k}$. Find a) $\text{proj}_{\mathbf{u}}(\mathbf{v}_1 + 2\mathbf{v}_2)$, b) $\text{proj}_{\mathbf{u}}3\mathbf{v}_1$ and c) $\text{proj}_{3\mathbf{u}}\mathbf{v}_2$.

9. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are three different vectors. Does the expression $\mathbf{v}_1 \cdot \mathbf{v}_2 \cdot \mathbf{v}_3$ make sense? What about $\mathbf{v}_1 \times \mathbf{v}_2 \times \mathbf{v}_3$? What about $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$? Is the last expression the same as $\mathbf{v}_1 \times (\mathbf{v}_2 \cdot \mathbf{v}_3)$, or the same as $(\mathbf{v}_1 \cdot \mathbf{v}_2) \times \mathbf{v}_3$?

10. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then does $\mathbf{v} = \mathbf{w}$?

11. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$

12. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$.

13. T/F If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then so is $\mathbf{a} \times \mathbf{b}$.

14. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$.

15. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$.

16. T/F If $\mathbf{a}, \mathbf{b}$ are two arbitrary vectors then $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{b} \times \mathbf{a}\|$.

17. What curve do you get by intersecting the sphere $x^2 + y^2 + z^2 = 4$ with the plane $x = y$?

18. Suppose a particle is moving with position vector $\mathbf{r}(t)$ and velocity vector $\mathbf{v}(t)$, and acceleration $\mathbf{a}(t)$. a) if $\mathbf{v}(t)$ is constant, must $|\mathbf{v}(t)|$ be constant? b) if $|\mathbf{v}(t)|$ is constant, must $\mathbf{v}(t)$ be constant? c) if $\mathbf{v}(t)$ and $\mathbf{r}(t)$ are orthogonal vectors, must $\mathbf{a}(t)$ and $\mathbf{v}(t)$ be orthogonal?

19. Find the intersection between the sphere $3x^2 + y^2 + z^2 = 4$ and the cylinder $y^2 + z^2 = 1$.

20. T/F Suppose that a GPS satellite is orbiting the Earth in such a way that its distance from the planet remains constant. Then the velocity vector of the satellite is always perpendicular to its position vector.

21. Suppose that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are two curves which intersect at a point P. If $P = \mathbf{r}_1(t_1)$ and $P = \mathbf{r}_2(t_2)$, does this mean that $t_1 = t_2$?

22. Suppose that $\mathbf{r}_1(t)$ is the equation of a curve and one defines $\mathbf{r}_2(t) = \mathbf{r}_1(\lambda t)$ for λ a positive constant (basically $\mathbf{r}_2$ is using a different time scale than the one used for $\mathbf{r}_1$). How is the velocity vector of $\mathbf{r}_2$ related to the one of $\mathbf{r}_1$? What about the acceleration vectors? If one defines $\mathbf{r}_3(t) = \mathbf{r}_1(t - t_0)$, how is the velocity vector of $\mathbf{r}_3$ related to the one of $\mathbf{r}_1$? What about the acceleration vectors?

23. Suppose $\mathbf{r}(t) = R\cos(\omega t)\mathbf{i} + R\sin(\omega t)\mathbf{j}$ gives the equation of a circle. What are the units of ω if t is given in units of seconds, or more generally, time? What values t must take if we want $\mathbf{r}(t)$ to give a parametrization of the circle? The period of $\mathbf{r}(t)$ is the smallest positive number T such that $\mathbf{r}(t+T) = \mathbf{r}(t)$. What is T in this situation?

24. Is the velocity vector $\mathbf{v}(t)$ of a curve $\mathbf{r}(t)$ always perpendicular to the acceleration vector $\mathbf{a}(t)$?

25. Find the point on the parabola $y = x^2$ which is closest to the point $(2, 1/2)$. ANS: $(1, 1)$

26. Let O represent the center of a circle, and A, B denote two points on the circle such that $\overrightarrow{AB}$ represents a diameter of the circle. If C is another point on the circle, show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are perpendicular vectors.

27. Show that the diagonals of a rhombus (a parallelogram with sides of equal length) are perpendicular.

28. Show that a parallelogram is a rectangle if and only if its diagonals have the same length.

29. Use the cross product to show that the area of a triangle with vertices $(0, 0)$, (a_1, a_2), (b_1, b_2) equals $\frac{1}{2}|a_1b_2 - a_2b_1|$.

30. Show that for any number k, the plane with equation $x - 2y + z + 3 + k(2x - y - z + 1) = 0$ contains the line of intersection of the planes $x - 2y + z + 3 = 0$ and $2x - y - z + 1 = 0$. Hint: there is a *really fast* way to do this!

31. Write a function of one variable which gives the distance between an arbitrary point on the line l with equation $ax + by = c$ and the point $P = (x_1, y_1)$. Then use this function to show that the distance from the line l to the point P is
$$d = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}} \tag{1}$$

32. The acceleration of a particle in the plane is $\mathbf{a} = 2\mathbf{i} + 8\mathbf{j}$. a) Find the particle's position as a function of t if its velocity at time $t = 0$ is $\mathbf{i} - \mathbf{j}$ and its position at time $t = 0$ is $2\mathbf{i} + 3\mathbf{j}$. b) Find the speed of the particle at time $t = 1$. ANS: $\mathbf{v}(t) = (2t + 1)\mathbf{i} + (8t - 1)\mathbf{j}$, $\mathbf{r}(t) = (t^2 + t + 2)\,\mathbf{i} + (4t^2 - t + 3)\,\mathbf{j}$. speed at $t = 1$ is $\sqrt{58}$.

33. Suppose that $\|\mathbf{v}\| = 7$, $\|\mathbf{w}\| = 6$, and the angle between $\mathbf{v}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$ radians. Compute $\|2\mathbf{v} - 3\mathbf{w}\|$. ANS: $2\sqrt{193}$

34. Simplify the following expression if $\|\mathbf{v}\| = 3$ and $\|\mathbf{w}\| = 4$: $(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) - 2\mathbf{v}\cdot\mathbf{w}$. ANS: 25

35. Find an equation for the curve which is the intersection of the plane $2x+4y+z = 4$ and the surface $z = x^2+y^2$. ANS: $\mathbf{r}(t) = (3\cos t - 1, 3\sin t - 2, (3\cos t - 1)^2 + (3\sin t - 2)^2)$

36. Consider $P = (1, 3)$, $Q = (5, -1)$, $R = (2, 3)$, $S = (x, 2)$ where x is unknown. a) Find $\overrightarrow{PQ}$ and $\overrightarrow{RS}$. b) Find the value of x so that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ become parallel. c) Find a unit vector in the direction of $\overrightarrow{RS}$. ANS: $\overrightarrow{PQ} = (4, -4)$, $\overrightarrow{RS} = (x-2, -1)$, $x = 3$, $\frac{(1,-1)}{\sqrt{2}}$.

2 Differential Calculus for Functions of Several Variables

1. Suppose that the level curves of a function $z = f(x, y)$ consists of straight lines. Must the graph of f be a plane?

2. Suppose that $T(x, y, z)$ represents the temperature at the point (x, y, z), measured in Kelvin. What are the units of $\frac{\partial T}{\partial x}$? What about the units of $\frac{\partial^2 T}{\partial x^2}$?

3. Suppose $f(x, y)$ is a function defined on the xy plane. Is it possible for $f(x, y)$ to have only local maxima but no local minima? What about the converse: can a function have only local minima but no local maxima? If a function has a critical point which is a saddle point, must it have critical points which are either local maxima or local minima?

4. If (a, b) is a critical point of f (in the sense that $f_x(a, b) = f_y(a, b) = 0$), is it true that (a, b) is a critical point of f^2? How about the converse? Namely, if (a, b) is a critical point of f^2, does that mean that (a, b) is a critical point of f?

5. True/False: if $\nabla f(a, b) = (0, 0)$, $f_{xx}(a, b) > 0$ and $f_{yy}(a, b) > 0$, then f has a local minimum at (a, b).

6. The Coulomb potential generated by a charge q located at the origin of the xy plane is $V(x, y) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2}}$, where ϵ_0 is a constant. a) What is the domain of V? b) What is the domain of V if it is considered as a function of x, y, z , namely, $V(x, y, z) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2}}$?

7. Suppose $T(x, y)$ represents the temperature of the floor at the point (x, y). If we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we can think of T as a function of r, θ. Find $\frac{\partial T}{\partial r}$ and $\frac{\partial T}{\partial \theta}$ in terms of $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$. We say that the temperature is *isotropic* if $\frac{\partial T}{\partial \theta} = 0$. Is the function $T(x, y) = \frac{y}{x}$ isotropic? What about $T(x, y) = x^2 + y^2$?

8. Suppose the domain of $f(x, y)$ consists of the rectangle $-1 \le x \le 1$, $0 \le y \le 1$, while the domain of $g(x, y)$ consists of the rectangle $0 \le x \le 1$, $-1 \le y \le 1$. What is the domain of $h(x, y) = f(x, y)g(x, y)$?

9. Suppose two level surfaces $f_1(x, y, z) = c_1$ and $f_2(x, y, z) = c_2$ intersect on a curve. What is an easy way to find the tangent vector $\mathbf{v}$ to any point on this curve from ∇f_1 and ∇f_2?

10. T/F The function $f(x, y) = (1 - x^2 - y^2)^{1/2}\ln(x^2 + y^2 - 1)$ has empty domain.

11. True/ False: If $\nabla f(x, y) = (0, 0)$, then (x, y) is a local minimum or local maximum of f

12. True/False: For any unit vector $\mathbf{u}$, $Df_{-\mathbf{u}}(\mathbf{r}) = -Df_{\mathbf{u}}(\mathbf{r})$

13. True/False: if $f(x, y) = \ln y$, then $\nabla f(x, y) = 1/y$

14. True/False: if f is differentiable at (a,b) and $\nabla f(a,b) = (0,0)$, then f has a local maximum or minimum at (a,b)

15. True/False: if $f(x,y)$ has two local maxima then f must have a local minimum

16. If $f(u,v,w)$ is a differentiable function and $u = x-y$, $v = y-z$ and $w = z-x$, show that
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0 \tag{2}$$

17. Let $f(x,y,z) = x^2+y^2+z^2$. At the point $(1,2,1)$, find the rate of change of f in the direction perpendicular to the plane $4x-4y+7z = 3$, pointing away from the origin. ANS = 2/3

18. Let $f(x,y,z)$ be a differentiable function and suppose that $\mathbf{r}(t)$ is a path which lies on the surface $f(x,y,z) = 17$. If $\mathbf{r}(5) = (1,4,2)$, show that $\mathbf{r}'(5)$ is orthogonal to $\nabla f(1,4,2)$.

19. For $f(x,y) = A - (x^2+Bx+y^2+Cy)$, what values of A, B, C give f a local maximum of 15 at the point $(-2,1)$? ANS: $A = 6$, $B = 4$, $C = -2$.

20. Let $\mathcal{S}$ denote the surface $x^2+y^2-3z^2 = 10$ and $P = (2,-3,1)$, which is a point on $\mathcal{S}$. Find an equation for the tangent plane to $\mathcal{S}$ at P. ANS: $-1 = \frac{2}{3}(x-2)-(y+3)-z$

21. A function $f(x,y)$ has gradient $(-3,4)$ at the point $P = (-1,3)$. Write an equation for the tangent line at P to the level curve of f passing through P.

22. Suppose that a duck is swimming in a circle, $x = \cos t$, $y = \sin t$, while the temperature is given by the formula $T = x^2e^y - xy^3$. Find $\frac{dT}{dt}$ using the chain rule. ANS: $\frac{dT}{dt} = \left(2\cos t e^{\sin t} - \sin^3 t\right)(-\sin t) + (\cos^2 t e^{\sin t} - 3\cos t \sin^2 t)(\cos t)$

23. Let $f(x,y) = \sqrt{(x-a)^2+(y-b)^2}$. Show that for any $(x,y) \neq (a,b)$, we have $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = 1$

24. Find the point on the graph of $z = 3x^2-4y^2$ at which the vector $\mathbf{n} = (2,3,1)$ is normal to the tangent plane. ANS: $\left(-\frac{1}{3}, \frac{3}{8}, -\frac{11}{48}\right)$

25. Find the directional derivative of
$$U(x,y,z) = 2x^3y - 3y^2z \tag{3}$$
at the point $P = (1,2,-1)$ in the direction toward the point $Q = (3,-1,5)$. ANS: $-\frac{90}{7}$

26. Use Lagrange multipliers to find the critical points of $f(x,y,z) = x+y+2z$, subject to the constraint $x^2+y^2+z^2 = 3$. ANS: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right)$

27. Find the rectangular box of maximum volume with edges of length x, y, z if the sum of the lengths of the edges $x+y+z$ is 132 cm. ANS: $x = y = z = 44$

28. Find the maximum volume of a box of rectangular base that must be inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. ANS: $\frac{8abc}{3\sqrt{3}}$

29. Find the shortest distance from the plane $3x - 2y - z = 3$ to the origin using Lagrange multipliers. ANS: $\frac{3}{\sqrt{14}}$

30. Use the Lagrange Multiplier methods to prove that the shortest distance from the point (a, b, c) to the plane $Ax + By + Cz + D = 0$ is $\left|\frac{Aa+Bb+Cc+D}{\sqrt{A^2+B^2+C^2}}\right|$.

3 Double and Triple Integrals

1. Suppose $f(x, y)$ and $g(x, y)$ are continuous functions on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Is it true that $\int_0^1 \int_0^1 f(x, y)g(x, y)dydx = \left(\int_0^1 \int_0^1 f(x, y)dydx\right)\left(\int_0^1 \int_0^1 g(x, y)dydx\right)$

2. Suppose $\rho(x, y)$ has units of electric charge per unit area. What are the units of $\int\int_R \rho(x, y)dA$? More generally, how are the units of $\int\int_R f(x, y)dA$ related to those of $f(x, y)$?

3. If $\int\int_R f(x, y)dA \geq 0$, does that mean that $f(x, y) \geq 0$ at every point of the region of integration R?

4. True/False: the integral $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\theta d\rho d\theta d\phi$ gives the volume of 1/4 of a sphere

5. True/False: For any $a, b \in \mathbb{R}$ and a continuous function f, $\int_0^a \int_0^b f(x, y)dydx = \int_0^b \int_0^a f(x, y)dxdy$

6. True/False: If $f(x, y) = g(x)h(y)$ then $\int\int_D f(x, y)dA = \left(\int\int_D g(x)dA\right)\left(\int\int_D h(y)dA\right)$

7. True/False: $\int_0^1 \int_0^x \sqrt{x + y^2}dydx = \int_0^x \int_0^1 \sqrt{x + y^2}dxdy$

8. True/False: For any integrable function f, $\int_0^a \int_y^a f(x, y)dxdy = \int_0^a \int_y^a f(x, y)dxdy$.

9. Integrate $f(x, y) = 6y + 16$ over the triangular region $\mathcal{T}$ with vertices $(0, 0)$, $(2, 4)$ and $(10, 0)$. ANS: 480

10. Sketch the region of integration and write an equivalent integral with the order of integration reversed for the integral $\int_0^2 \int_{x^2}^{2x} xydydx$. Evaluate the integral in both forms. ANS: $\int_0^4 \int_{y/2}^{\sqrt{y}} xydxdy = 8/3$

11. Evaluate $\int_0^1 \int_y^1 \sin(x^2)dxdy$. ANS $(1 - \cos(1))/2$

12. Use a double integral in polar coordinates to calculate the area of the polygonal region with vertices $(0,0)$, $(B,0)$ and (B,H). ANS: $\frac{1}{2}BH$

13. Let E denote the region in the first octant that is bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2=9$. Express the volume of E as an iterated triple integral in i) cylindrical and ii) spherical coordinates. Then evaluate both integrals. ANS$=9\pi(2-\sqrt{2})/4$

14. Let E denote the region inside the sphere $x^2+y^2+z^2=25$ and inside the cylinder $x^2+y^2=16$. ANS$=392\pi/3$

15. Let $\mathcal{D}$ be the region in the first quadrant bounded by the curves $y=x$, $y=3x$, $xy=1$ and $xy=3$. Evaluate $\int\int_D xydA$ using the transformation $x=u/v$, $y=v$. ANS$=2\ln 3$

16. Calculate the volume of the region $\mathcal{W}$ inside both spheres $x^2+y^2+z^2=4$ and $x^2+y^2+(z-2)^2=4$. ANS: $10\pi/3$.

17. Let $\mathcal{E}$ be the region inside the ellipse $2x^2+2xy+5y^2=4$. Noting that $2x^2+2xy+5y^2=(x+2y)^2+(x-y)^2$, apply the change of variables $u=x+2y$, $v=x-y$ in order to find the area of $\mathcal{E}$. ANS$=4\pi/3$

18. Find the bounds for the region R determined by the curves $y=x^2$ and $y=x+2$ and write $\int\int_R f(x,y)dA$ in two different ways. ANS: $\int_{-1}^{2}\int_{x^2}^{x+2} f(x,y)dydx$, $\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dxdy+\int_1^4\int_{y-2}^{\sqrt{y}} f(x,y)dxdy$

19. Find the bounds for the region R determined by the curves $y=e^x$, $y=e^{-x}$, the vertical lines $x=-2$ and $x=1$. Write $\int\int_R f(x,y)dA$ in the order $dydx$. ANS: $\int_{-2}^{0}\int_0^{e^x} f(x,y)dydx+\int_0^1\int_0^{e^{-x}} f(x,y)dydx$

20. Find the bounds for the region R in the first quadrant determined by the curves $x^2+y^2=4$ and $y=-x+2$. Write $\int\int_R f(x,y)dA$ in the order $dxdy$. ANS: $\int_0^2\int_{2-y}^{\sqrt{4-y^2}} f(x,y)dxdy$

21. Find $\int_0^2\int_{x^2}^4 x^3e^{y^3}\,dy\,dx$. ANS: $\frac{1}{12}(e^{64}-1)$

22. Find $\int_0^{1/2}\int_{\sqrt{3}x}^{\sqrt{1-x^2}} 9xdydx$. ANS: $3\left(1-\frac{\sqrt{3}}{2}\right)$

23. For $a>0$, find the double integral

$$\int_0^a\int_x^{\sqrt{2a^2-x^2}} xy(x+y)dydx \tag{4}$$

by switching to polar coordinates. Your answer will depend on a. ANS: $\frac{4\sqrt{2}a^5}{15}$

24. The integral $\int_0^{\pi}\int_{2\sqrt{2}}^{4}\int_0^{\sqrt{16-r^2}}(16r-r^3-z^2r)dzdrd\theta$ is given in cylindrical coordinates. Write (but do not compute) the integral using spherical coordinates. ANS: $\int_0^{\pi}\int_{\pi/4}^{\pi/2}\int_{\frac{2\sqrt{2}}{\sin\phi}}^{4}(16-\rho^2)\rho^2\sin\phi d\rho d\phi d\theta$

25. Write the double integral $\int_1^2\int_0^{\ln x}(x-1)\sqrt{1+e^{2y}}dydx$ in the order $dxdy$. Do not find the value. ANS: $\int_0^{\ln 2}\int_{e^y}^{2}(x-1)\sqrt{1+e^{2y}}dxdy$

26. $\int_0^a\int_0^{\sqrt{ax-x^2}}\frac{adydx}{\sqrt{a^2-x^2-y^2}}$. ANS: $a^2\left(\frac{\pi}{2}-1\right)$

27. Use spherical coordinates to evaluate $\int\int\int_T \frac{dxdydz}{(x^2+y^2+z^2)^{3/2}}$, where T is the region bounded by the spheres $x^2+y^2+z^2=4$, $x^2+y^2+z^2=9$ and the half-cone $x^2+y^2-z^2=0$, $z\geq 0$. ANS: $2\pi\left(1-\frac{1}{\sqrt{2}}\right)\ln\left(\frac{3}{2}\right)$

4 Vector Fields

1. True/False: $\nabla\cdot(\nabla\times\mathbf{F})=0$

2. True/False: If $\mathbf{F},\mathbf{G}$ are vector fields and $\nabla\times\mathbf{F}=\nabla\times\mathbf{G}$, then $\mathbf{F}=\mathbf{G}$.

3. True/False: If $\mathbf{F}$ is conservative then $\nabla\cdot\mathbf{F}=0$

4. True/False: curl(div$\mathbf{F}$)) is not a meaningful expression

5. Let $\mathcal{D}$ be the triangle in the xy plane with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. a) Evaluate $\int 2y^2dx+2xdy$ as a line integral. b) Evaluate the line integral in a) by using Green's Theorem and then evaluating a double integral. ANS= 1/3

6. Show that the vector field $\mathbf{F}=(3x^2y^2z+1, 2x^3yz+2, x^3y^2+3)$ has zero curl. Then find a potential V with $\nabla V=\mathbf{F}$. ANS: $x^3y^2z+x+2y+3z$

7. Evaluate $\int_C(y+e^{x^2})dx-xdy$, where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, oriented counterclockwise. ANS -2

8. Calculate the flux $\int\int_{\mathcal{S}}\mathbf{F}\cdot\mathbf{n}d\sigma$ where $\mathcal{S}$ is the part of the plane $2x+5y+z=10$ in the first octant with the upward-pointing normal, and $\mathbf{F}=<z,2y,6x>$. ANS: 5950/3

9. Calculate $\int\int_{\mathcal{S}}\mathbf{F}\cdot\mathbf{n}d\sigma$, where $\mathcal{S}$ is the part of the paraboloid $z=9-x^2-y^2$ on or above the plane $z=1$, with the upward-pointing normal, and $\mathbf{F}=\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}},z\right)$

10. Find the work of the vector field $\mathbf{F}(x,y,z)=y\mathbf{i}$ along the curve which is obtained as the intersection of the surfaces $z=x^2+y^2-2$ and $2x+4y=z+2$. **Hint:** you may find it useful to complete the squares and use the identity $\sin^2 t=\frac{1}{2}(1-\cos(2t))$. ANS: $-\pi$

11. Consider the vector field $\mathbf{F}(x,y,z) = (y+y^2z)\mathbf{i} + (x-z+2xyz)\mathbf{j} + (-y+xy^2)\mathbf{k}$. Compute the line integral (work)$\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$, C_1 is the curve $\mathbf{r}(t)$ with equation $\mathbf{r}(t) = (\cos t, t-1, e^{1-\sin t})$ and $0 \le t \le \pi/2$. ANS: $-\frac{\pi}{2} - 2e + 2$

12. Evaluate the work (line integral)$\int_C ydx + xdy + 2zdz$. Here C is the curve with equation $\mathbf{r}(t) = t(t-1)e^{\sqrt{t}}\mathbf{i} + \sin\left(\frac{\pi}{2}t^2\right)\mathbf{j} + \frac{t}{t^2+1}\mathbf{k}$ for $0 \le t \le 1$. ANS: $\frac{1}{4}$

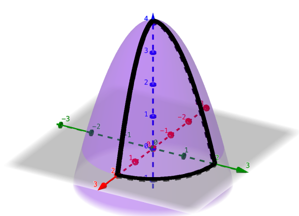
13. Find the value of the constant λ for which the vector field $\mathbf{F}(x,y,z) = (2xy + \lambda z^2, 2yz + \lambda x^2, 2xz + \lambda y^2)$ is conservative. ANS: $\lambda = 1$

5 Divergence and Stokes Theorem

1. Recall that if $\mathbf{a}, \mathbf{b}$ are two vectors then $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0$, since $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$ so their dot product must be zero. Likewise, we have that $\mathbf{a}\times\mathbf{a} = \mathbf{0}$. There are analogues of these identities which involve the entity ∇. More precisely, a) show that if $\mathbf{F}(x,y,z)$ is a vector field, then $\nabla\cdot(\nabla\times\mathbf{F}) = 0$ and b) if V is an arbitrary scalar function, then $\nabla\times(\nabla V) = \mathbf{0}$.

2. Use Stokes' theorem to find the work $\oint \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = z\mathbf{i} + y\mathbf{k}$ and C is the unit circle $x^2 + y^2 = R^2$. ANS: 0

3. Consider the closed curve C obtained from the intersection of the paraboloid $z = 4 - x^2 - y^2$with the three octants $x = 0$, $y = 0$, $z = 0$, as shown in the figure (so the curve consists of three pieces). Use Stokes' theorem to compute the line integral (work)$\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z) = (yz)\mathbf{i} - xz\mathbf{j} + \mathbf{k}$. When applying Stokes' theorem, choose the normal vector $\mathbf{n}$ on the surface S which has positive third



coordinate. ANS: 0

4. Using Stokes' theorem, find $I = \int\int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}d\sigma$, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + (y+z)\mathbf{k}$ and S is the portion of the surface $2x + y + z = 2$ above the first octant and $\mathbf{n}$ is the unitary normal vector to the surface, with non-negative z component. ANS: 2

5. Use Stokes' theorem to evaluate $\int_C -y^3dx + x^3dy - z^3dz$ where C is the intersection of the cylinder $x^2 + y^2 = 1$, and the plane $x + y + z = 1$. Assume C is oriented counterclockwise with respect to the xy plane. ANS: $\frac{6\pi}{4}$

6. Verify Stokes' theorem for the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$, the its boundary $x^2 + y^2 = 9, z = 0$ and the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$. ANS: -18π

7. For $\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S the unit sphere $x^2 + y^2 + z^2 = 1$, compute $\int \mathbf{F} \cdot \mathbf{n} d\sigma$ using the divergence theorem. ANS: $8\pi/3$

8. Verify the divergence theorem for $\mathbf{F} = 2x^2y\mathbf{i} - y^2\mathbf{j} + 4xz^2\mathbf{k}$ and the region bounded by $y^2 + z^2 = 9$, $x = 0$ and $x = 2$. ANS: 0

9. Find the divergence of $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and $\mathbf{G} = \frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}$. ANS: 2 and 0 respectively.