MATH 251, Mariano Echeverria

Curves and Geometry in Space

This material corresponds roughly to sections 12.5, 13.1, 13.2, 13.3 and 13.4 in the book.

Equation of a line
The equation of a line l passing through point P and with direction specified by the vector $\mathbf{v}$ is given by the (parametric) equation

$$\mathbf{r}(t) = P + t\mathbf{v} \tag{1}$$

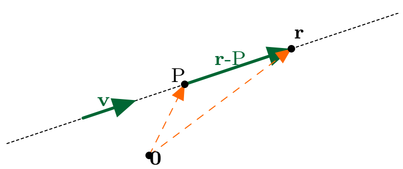

Figure 1: Equation of a line through a point P and with direction vector $\mathbf{v}$

Example 1. Consider the lines L_1, L_2 with equations

$$L_1 : (x, y, z) = (t + 2, -t + 4, 2t + 6) \tag{2}$$

$$L_2 : (x, y, z) = (-t + 1, t + 5.t + 7) \tag{3}$$

a) Determine the point Q where the lines intersect.
Write Q as $Q = (q_1, q_2, q_3)$. Since Q belongs to L_1 there is a time t_1 for which

$$(q_1, q_2, q_3) = (t_1 + 2, -t_1 + 4, 2t_1 + 6) \tag{4}$$

Likewise, Q is in L_2 so there is a time t_2 such that

$$(q_1, q_2, q_3) = (-t_2 + 1, t_2 + 5, t_2 + 7) \tag{5}$$

Hence we obtain the system of equations

$$\begin{cases} t_1 + t_2 = -1 \\ -t_1 - t_2 = 1 \\ 2t_1 - t_2 = 1 \end{cases} \tag{6}$$

Adding the first and third equations we conclude that $t_1 = 0$ and substituting in the first equation we find that $t_2 = -1$. Using equation (4) we conclude that the point Q is

$$(q_1, q_2, q_3) = (2, 4, 6) \tag{7}$$

b) Check that if A belongs to the line L_1 and B belongs to the line L_2 then $\overrightarrow{QA}$ and $\overrightarrow{QB}$ are orthogonal vectors.

If $A \in L_1$ then there exists t_1 such that $A = (t_1 + 2, -t_1 + 4, 2t_1 + 6)$ and if $B \in L_2$ there exists t_2 such that $B = (-t_2 + 1, t_2 + 5, t_2 + 7)$

$$\overrightarrow{QA} = A - Q = (t_1 + 2, -t_1 + 4, 2t_1 + 6) - (2, 4, 6) = (t_1, -t_1, 2t_1) \tag{8}$$

$$\overrightarrow{QB} = B - Q = (-t_2 + 1, t_2 + 5, t_2 + 7) - (2, 4, 6) = (-t_2 - 1, t_2 + 1, t_2 + 1) \tag{9}$$

so

$$\overrightarrow{QA} \cdot \overrightarrow{QB} = (t_1, -t_1, 2t_1) \cdot (-t_2 - 1, t_2 + 1, t_2 + 1) = t_1(-t_2 - 1) - t_1(t_2 + 1) + 2t_1(t_2 + 1) \tag{10}$$

$$= -2t_1(t_2 + 1) + 2t_1(t_2 + 1) = 0 \tag{11}$$

c) Let $C = (3, 3, 8)$. Check that C belongs to the line L_1.

Just take $t = 1$ for the equation $L_1 : (x, y, z) = (t + 2, -t + 4, 2t + 6)$.

d) Find a point D on the line L_2 such that the area of the triangle CQD equals $\sqrt{18}$.

We already know that C and Q belong to L_1 and the vector $\overrightarrow{CQ} = Q - C = (2, 4, 6) - (3, 3, 8) = (-1, 1, -2)$ has norm $\sqrt{6}$. This vector serves as the base of the triangle CQD.

Since Q also belongs to L_2 and the lines are orthogonal because of part $b)$ then this triangle must be a right triangle and the vector representing the height can taken to be $\overrightarrow{DQ} = (2, 4, 6) - (-t + 1, t + 5, t + 7) = (t + 1, -t - 1, -t - 1)$, which has norm $\sqrt{3}\,|t + 1|$. Therefore we need to solve the equation

$$\sqrt{18} = \frac{1}{2}\sqrt{6}\sqrt{3}\,|t + 1| \tag{12}$$

which has solutions $t = 1$ or $t = -3$. Substituting $t = 1$ in the line for L_2 gives the point $(0, 6, 8)$. Notice that $t = -3$ would have also worked.

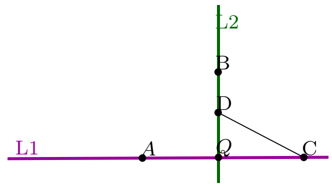

Figure 2: Perpendicular lines

Example 2. Find the equation for the line L_1 which goes through the points $P = (4, 6, 7)$ and $Q = (3, 6, 9)$.

We can take $\overrightarrow{PQ}$ as the direction vector for the line L_1:

$$\overrightarrow{PQ} = Q - P = (3, 6, 9) - (4, 6, 7) = (-1, 0, 2) \tag{13}$$

Therefore the equation for the line can be taken to be

$$L_1 : \mathbf{r} = (x, y, z) = P + t\overrightarrow{PQ} = (4, 6, 7) + t(-1, 0, 2) \tag{14}$$

Equation of a plane

The **(parametric) equation** of a plane passing through the point P and with direction vectors $\mathbf{u}, \mathbf{v}$ is

$$\mathbf{r}(t) = P + t\mathbf{u} + s\mathbf{v} \tag{15}$$

Another way to describe a plane is by specifying a vector $\mathbf{n}$ perpendicular to the plane. In this case, if $\mathbf{r} = (x, y, z)$ denotes an arbitrary point on the plane, the **normal equation** is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_P) = 0 \tag{16}$$

If we write $\mathbf{n}$ as $\mathbf{n} = (a, b, c)$ and define $d = \mathbf{n} \cdot \mathbf{r}_P$, then you will see the previous equation written as

$$ax + by + cz = d \tag{17}$$

The important point of this equation is that the vector $\mathbf{n}$ can be read from the coefficients multiplying each of the variables x, y, z on the left hand side.

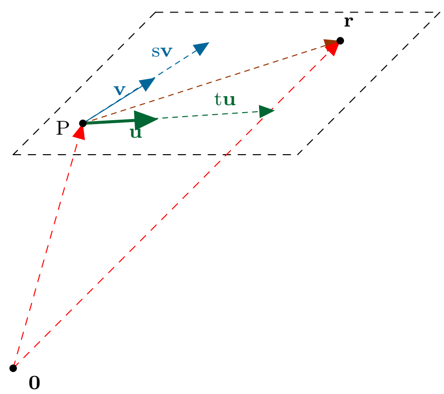

Figure 3: Vector equation for a plane

Suppose two planes have equations $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$. If we want to determine the intersection of the two planes (which unless they are the same or parallel will correspond to a line), we need to solve the two equations

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases} \tag{18}$$

For example, to find the intersection between the planes $x + 3y - 5z = 2$ and $2x - 3z = 0$ we have to solve

$$\begin{cases} x + 3y - 5z = 2 \\ 2x - 3z = 0 \end{cases} \tag{19}$$

which can be simplified to [substitute $x = (3/2)z$ into the first equation]

$$\begin{cases} y - \frac{7}{6}z = \frac{2}{3} \\ x - \frac{3}{2}z = 0 \end{cases} \tag{20}$$

Taking $z = t$ we can write

$$(x, y, z) = \left(\frac{3}{2}t, \frac{7}{6}t + \frac{2}{3}, t\right) = \left(0, \frac{2}{3}, 0\right) + t\left(\frac{3}{2}, \frac{7}{6}, 1\right) \tag{21}$$

which is the vector equation for a line.

Velocity of a particle:

- If $\mathbf{r}(t)$ denotes the position vector of a particle at time t, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} \tag{22}$$

  can be interpreted as the **velocity** of the particle at time t. It will be a vector tangent to the curve at that point, thus generalizing the concept of the derivative as the slope of the tangent line.

- In practice, if we write $\mathbf{r}(t)$ as

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{23}$$

  Then the velocity can be computed as

$$\mathbf{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{24}$$

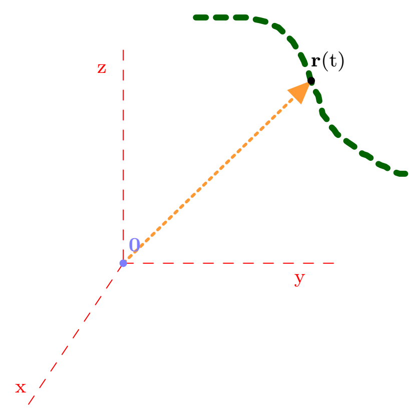

Figure 4: Motion of a particle in space

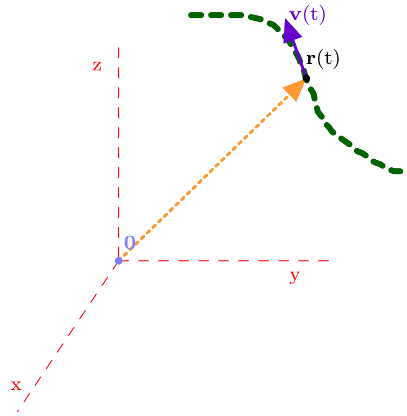

Figure 5: Velocity vector is tangent to the curve

Product rules for time dependent vectors

If $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are time dependent vectors and $f(t)$ is a time dependent scalar function, then

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \left(\frac{d}{dt}\mathbf{a}\right)\cdot\mathbf{b} + \mathbf{a}\cdot\left(\frac{d}{dt}\mathbf{b}\right) \tag{25}$$

$$\frac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \left(\frac{d}{dt}\mathbf{a}\right)\times\mathbf{b} + \mathbf{a}\times\left(\frac{d}{dt}\mathbf{b}\right) \tag{26}$$

$$\frac{d}{dt}(f\mathbf{a}) = \left(\frac{d}{dt}f\right)\mathbf{a} + f\frac{d}{dt}\mathbf{a} \tag{27}$$

As a consequence of the first product rule, if $\mathbf{v}(t)$ is a vector of constant norm, then its derivative $\frac{d}{dt}\mathbf{v}(t)$ is orthogonal to $\mathbf{v}(t)$ at every time, that is,

$$\mathbf{v}(t)\cdot\frac{d}{dt}\mathbf{v}(t) = 0 \tag{28}$$

Frenet Moving Frame [this topic will not be evaluated on the exam, only on the written homework ☺]

- From now on assume that the curve $\mathbf{r}(t)$ is such that at any instant the velocity is non-vanishing, that is,

- $$\mathbf{v}(t) \neq \mathbf{0} \tag{29}$$

  Such curves are known sometimes as **regular curves,** or more precisely, a regular parameterization of the curve. We can therefore decompose the velocity as

- $$\mathbf{v}(t) = |\mathbf{v}(t)| \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = |\mathbf{v}(t)| \, \mathbf{T}(t) \tag{30}$$

  where
  $$\mathbf{T}(t) \equiv \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} \tag{31}$$
  is the **tangent vector to the curve.** Notice that it is a unitary vector.

- Since the norm of $\mathbf{T}(t)$ is time independent, $\frac{d}{dt}\mathbf{T}(t)$ will always be orthogonal to $\mathbf{T}(t)$. Thus we define the **normal vector**
  $$\mathbf{N}(t) \equiv \frac{\frac{d}{dt}\mathbf{T}(t)}{\left|\frac{d}{dt}\mathbf{T}(t)\right|} \tag{32}$$

- Finally, the **binormal vector** is defined as
  $$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \tag{33}$$
  It is also another unitary vector.

- The **Frenet moving frame** to the curve $\mathbf{r}(t)$ is the system of vectors $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$. It is the moving analogue of the system $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Example 3. The equation of a particle moving on a circle of radius R with angular velocity ω is
$$\mathbf{r}(t) = R\cos\omega t\mathbf{i} + R\sin\omega t\mathbf{j} \tag{34}$$
Its velocity is
$$\mathbf{v}(t) = -R\omega\sin\omega t\mathbf{i} + R\omega\cos\omega t\mathbf{j} \tag{35}$$
and the acceleration is

$$\mathbf{a}(t) = -R\omega^2\cos\omega t - R\omega^2\sin\omega t\mathbf{j} \tag{36}$$

To find the tangent vector notice that the norm of the velocity is
$$|\mathbf{v}(t)| = R\omega \tag{37}$$

so

$$\mathbf{T}(t) = -\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j} \tag{38}$$

The derivative of the tangent vector is

$$\frac{d}{dt}\mathbf{T}(t) = -\omega\cos\omega t\mathbf{i} - \omega\sin\omega t\mathbf{j} \tag{39}$$

and it has norm

$$\left|\frac{d}{dt}\mathbf{T}(t)\right| = \omega \tag{40}$$

Therefore the normal vector is

$$\mathbf{N}(t) = -\cos\omega t\mathbf{i} - \sin\omega t\mathbf{j} \tag{41}$$

while the binormal vector is

$$\mathbf{B}(t) = (-\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j}) \times (-\cos\omega t\mathbf{i} - \sin\omega t\mathbf{j}) = \mathbf{k} \tag{42}$$

Therefore

$$\begin{cases} \mathbf{T}(t) = -\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j} \\ \mathbf{N}(t) = -\cos\omega t\mathbf{i} - \sin\omega t\mathbf{j} \\ \mathbf{B}(t) = \mathbf{k} \end{cases} \tag{43}$$

Notice that $\mathbf{N}(t)$ has the same direction as the centripetal acceleration, while $\mathbf{B}(t)$ is parallel to the angular velocity.

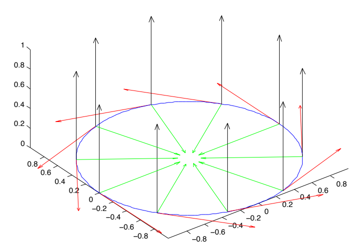

Figure 6: Circle: $\mathbf{T}(t)$ is red , $\mathbf{N}(t)$ is green and $\mathbf{B}(t)$ is black

Example 4. Another classical example is the helix

$$\mathbf{r}(t) = a\cos(t)\,\mathbf{i} + a\sin(t)\,\mathbf{j} + bt\mathbf{k} \tag{44}$$

where $a > 0$. The velocity is

$$\mathbf{v}(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k} \tag{45}$$

and the acceleration is

$$\mathbf{a}(t) = -a\cos t\mathbf{i} - a\sin t\mathbf{j} \tag{46}$$

The magnitude of the velocity is

$$|\mathbf{v}(t)| = \sqrt{a^2 + b^2} \tag{47}$$

so the tangent vector is

$$\mathbf{T}(t) = -\frac{a\sin t}{\sqrt{a^2+b^2}}\mathbf{i} + \frac{a\cos t}{\sqrt{a^2+b^2}}\mathbf{j} + \frac{b}{\sqrt{a^2+b^2}}\mathbf{k} \tag{48}$$

The derivative is also easy to compute

$$\frac{d}{dt}\mathbf{T}(t) = -\frac{a\cos t}{\sqrt{a^2+b^2}}\mathbf{i} - \frac{a\sin t}{\sqrt{a^2+b^2}}\mathbf{j} \tag{49}$$

as well as its norm

$$\left|\frac{d}{dt}\mathbf{T}(t)\right| = \frac{a}{\sqrt{a^2+b^2}} \tag{50}$$

Thus the normal vector is

$$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} \tag{51}$$

while the binormal vector is

$$\begin{aligned}\mathbf{B}(t) &= \left(-\tfrac{a\sin t}{\sqrt{a^2+b^2}}\mathbf{i} + \tfrac{a\cos t}{\sqrt{a^2+b^2}}\mathbf{j} + \tfrac{b}{\sqrt{a^2+b^2}}\mathbf{k}\right) \times (-\cos t\mathbf{i} - \sin t\mathbf{j}) \\ &= \tfrac{b}{\sqrt{a^2+b^2}}\sin t\mathbf{i} - \tfrac{b}{\sqrt{a^2+b^2}}\cos t\mathbf{j} + \tfrac{a}{\sqrt{a^2+b^2}}\mathbf{k}\end{aligned} \tag{52}$$

Therefore we just found the Frenet frame for the helix.

$$\begin{cases}\mathbf{T}(t) = -\tfrac{a\sin t}{\sqrt{a^2+b^2}}\mathbf{i} + \tfrac{a\cos t}{\sqrt{a^2+b^2}}\mathbf{j} + \tfrac{b}{\sqrt{a^2+b^2}}\mathbf{k} \\ \mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} \\ \mathbf{B}(t) = \tfrac{b}{\sqrt{a^2+b^2}}\sin t\mathbf{i} - \tfrac{b}{\sqrt{a^2+b^2}}\cos t\mathbf{j} + \tfrac{a}{\sqrt{a^2+b^2}}\mathbf{k}\end{cases} \tag{53}$$

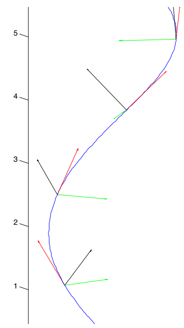

Figure 7: Helix: $\mathbf{T}(t)$ is red , $\mathbf{N}(t)$ is green and $\mathbf{B}(t)$ is black

Arc Length and length of a curve [this topic will not be evaluated on the exam, only on the written homework ☺]

- Let v denote the **speed of the curve**

$$v = |\mathbf{v}| \tag{54}$$

- We define the arc length as

$$s(t) = \int_{t_0}^{t} v(\tau) d\tau \tag{55}$$

or infinitesimally

$$ds = v(t)dt \tag{56}$$

The main property or arc length is that if we write the curve $\mathbf{r}(t)$ as a function of s instead, that is, we consider it as $\mathbf{r}(s)$, then the particle travels at unit speed with respect to s. In other words

$$\|\frac{d\mathbf{r}}{ds}\| = 1 \tag{57}$$

- If the curve has length L from point $\mathbf{r}(t_1)$ to the point $\mathbf{r}(t_2)$ then it can be computed as

$$L \equiv \int_0^L ds = \int_{t_0}^{t_1} \frac{ds}{dt} dt = \int_{t_0}^{t_1} v dt \tag{58}$$

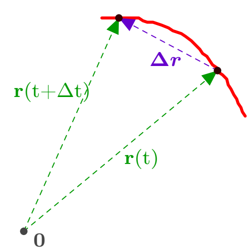

Figure 8: Length of a curve

Example 5. Compute the length of a circle using the previous example as well as a parameterization of the equation in terms of the arc length.

We had already found that

$$\mathbf{v}(t) = -R\omega \sin \omega t \mathbf{i} + R\omega \cos \omega t \mathbf{j} \tag{59}$$

The speed v is simply the norm of $\mathbf{v}$, namely

$$v = R\omega \tag{60}$$

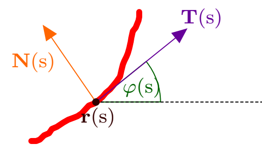

Figure 9: Curvature

Since $\omega = \frac{2\pi}{T}$ where T is the period, the time it takes to travel the circle is the interval $\left[0, \frac{2\pi}{\omega}\right]$ so the total length is

$$L = \int_0^{\frac{2\pi}{\omega}} R\omega dt = 2\pi R \tag{61}$$

The arc length s can be computed as

$$s = \int_0^t v(\tau)d\tau = \int_0^t R\omega d\tau = R\omega t \tag{62}$$

This allows us to write t as a function of s

$$t = \frac{s}{R\omega} \tag{63}$$

Using the old parameterization for the circle

$$\mathbf{r}(t) = R\cos\omega t\mathbf{i} + R\sin\omega t\mathbf{j} \tag{64}$$

and replacing t as a function of s we find the desired new parameterization

$$\mathbf{r}(s) = R\cos\left(\frac{s}{R}\right)\mathbf{i} + R\sin\left(\frac{s}{R}\right)\mathbf{j} \tag{65}$$

In order to define the concept of curvature suppose that the curve $\mathbf{r}(s)$ is being parameterized by arc length and lies on the xy plane. If $\varphi(s)$ is the angle between the tangent vector and the x axis, the change in direction of the curve is given by $\frac{d}{ds}\varphi(s)$ so this suggests we should define the **signed curvature** $\kappa_r(s)$ as

$$\kappa_r(s) \equiv \varphi'(s) \tag{66}$$

On the other hand, by the definition of the dot product and the fact that the tangent vector is unitary we also have

$$\cos\varphi(s) = \mathbf{T}(s)\cdot\mathbf{i} \tag{67}$$

so differentiating with respect to s

$$-\sin\varphi(s)\,\kappa_r(s) = \mathbf{T}'(s)\cdot\mathbf{i} = \left|\mathbf{T}'(s)\right|\mathbf{N}(s)\cdot\mathbf{i} \tag{68}$$

Given that the normal vector is always orthogonal to the tangent vector, the angle between $\mathbf{N}$ and the x axis is $\varphi(s) - \frac{\pi}{2}$ or $\frac{\pi}{2} - \varphi(s)$ depending on the direction of the normal vector. Since cosine is an even function and the normal vector is unitary, we have

$$\mathbf{N}(s) \cdot \mathbf{i} = \cos\left(\varphi(s) \pm \frac{\pi}{2}\right) = \mp \sin\left(\varphi(s)\right) \tag{69}$$

thus substituting this into 68 we find that

$$\kappa_r(s) = \pm \left|\mathbf{T}'(s)\right| \tag{70}$$

We only care for our purposes about the absolute value of the curvature, so we define

[this topic will not be evaluated on the exam, only on the written homework ☺]
The **curvature** of a curve $\mathbf{r}(s)$ parameterized by arc length is

$$\kappa(s) \equiv \left|\mathbf{T}'(s)\right| \tag{71}$$

It relates $\mathbf{T}'(s)$ and $\mathbf{N}(s)$ as

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s) \tag{72}$$

Example 6. Compute the curvature of a straight line
We can parameterize the line as

$$\mathbf{r}(s) = \mathbf{r}(0) + s\boldsymbol{v} \tag{73}$$

where $\mathbf{v}$ is a unitary vector giving the direction of the line. Hence

$$\mathbf{T}(s) = \mathbf{r}'(s) = \mathbf{v} \tag{74}$$

so

$$\mathbf{T}'(s) = 0 \tag{75}$$

which means that

$$\kappa(s) = 0 \tag{76}$$

Thus straight lines are not curved, as expected!
In fact, if the curvature is always vanishing, then from the equation 71 we can deduce that $\mathbf{T}(s) = \mathbf{T}(0)$ is independent of s. Since

$$\frac{d\mathbf{r}}{ds} = \mathbf{T}(s) = \mathbf{T}(0) \tag{77}$$

integrating with respect to s we find that

$$\mathbf{r}(s) = \mathbf{T}(0)s + \mathbf{r}(0) \tag{78}$$

In other words

A curve $\mathbf{r}(s)$ has a curvature function $\kappa(s)$ which vanishes identically if and only

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Example 7. Find the curvature of a circle.
We can use the parameterization found before in terms of arc length

$$\mathbf{T}'(s) = -\frac{1}{R}\cos\left(\frac{s}{R}\right)\mathbf{i} - \frac{1}{R}\sin\left(\frac{s}{R}\right)\mathbf{j} \tag{79}$$

and the curvature is

$$\kappa(s) = \left|\mathbf{T}'(s)\right| = \frac{1}{R} \tag{80}$$

Notice that the curvature is independent of the point on the circle and as the circle increases in size, moving along the circle "feels" less curved. From this point of view, a straight line can be regarded as a circle of infinite radius.

Finally, we discuss an interesting way to decompose the acceleration. We start with the formula

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \frac{d}{dt}\left(|\mathbf{v}(t)|\,\mathbf{T}(t)\right) = \frac{d}{dt}\left(v\mathbf{T}(t)\right) \tag{81}$$

And use the product rule to differentiate this

$$\mathbf{a}(t) = \frac{dv}{dt}\,\mathbf{T}(t) + v\frac{d}{dt}\mathbf{T}(t) = \frac{dv}{dt}\,\mathbf{T}(t) + v\left|\frac{d}{dt}\mathbf{T}(t)\right|\mathbf{N}(t) \tag{82}$$

Now this is the same as

$$\mathbf{a}(t) = \frac{dv}{dt}\,\mathbf{T}(t) + v^2\kappa(t)\mathbf{N}(t) \tag{83}$$

Therefore the acceleration always belong to the plane determined by **T** y **N**, which is called the **osculating plane.**

As a final computation, if we want to know the curvature with respect to an arbitrary parameter, notice that

$$\begin{aligned} |\mathbf{v}(t) \times \mathbf{a}(t)| &= |v\mathbf{T}(t) \times (\mathbf{a}_{\mathbf{T}}(t) + \mathbf{a}_{\mathbf{N}}(t))| = |v\mathbf{T}(t) \times \mathbf{a}_N(t)| \\ &= v^3\kappa(t) \end{aligned} \tag{84}$$

Thus

$$\kappa(t) = \frac{1}{v^3}\,|\mathbf{v}(t) \times \mathbf{a}(t)| \tag{85}$$

[this topic will not be evaluated on the exam, only on the written homework ☺]

- The **tangential acceleration** is

$$\mathbf{a_T}(t) \equiv \frac{dv}{dt}\mathbf{T}(t) \tag{86}$$

- The **normal acceleration** is

$$\mathbf{a_N}(\mathbf{t}) = v^2\kappa(t)\mathbf{N}(t) \tag{87}$$

- To compute the curvature with respect to an arbitrary parameter we can use the formulas

$$\begin{aligned} \kappa(t) &= \frac{1}{v}\left|\frac{d}{dt}\mathbf{T}(t)\right| \\ \kappa(t) &= \frac{1}{v^3}\left|\mathbf{v}(t) \times \mathbf{a}(t)\right| \end{aligned} \tag{88}$$