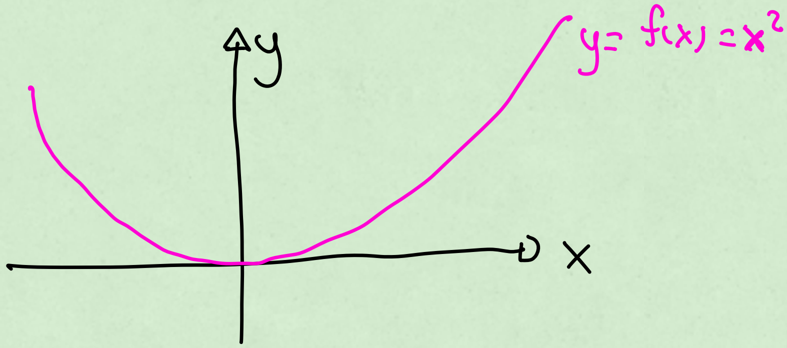
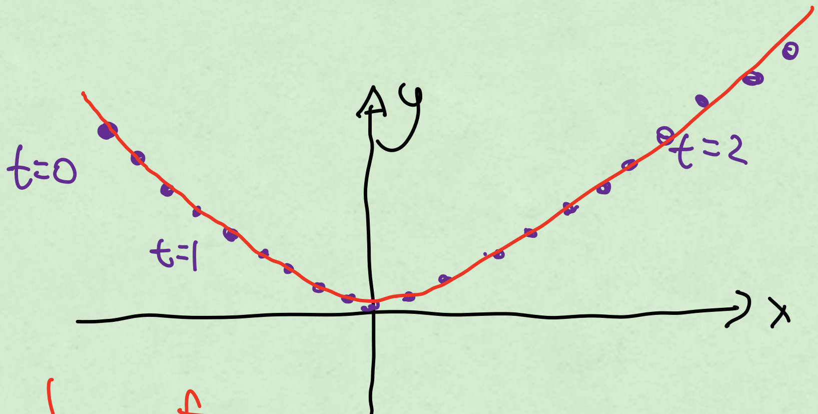
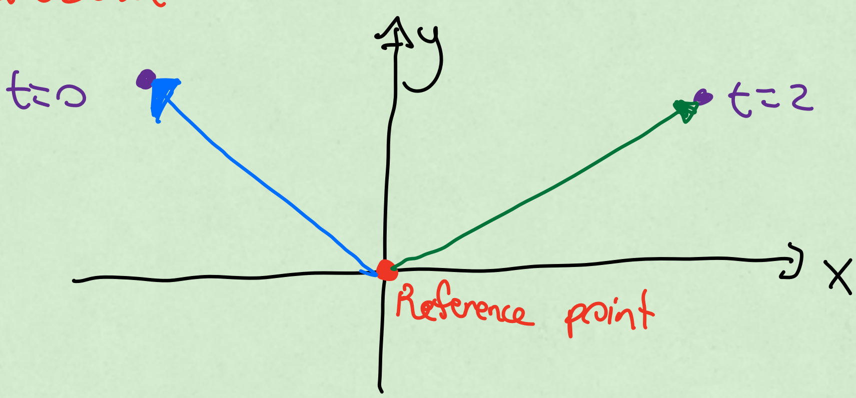
Lecture 1 [12.1 - 12.2]

graph
= a curve
on the xy
plane
[static object]
painting



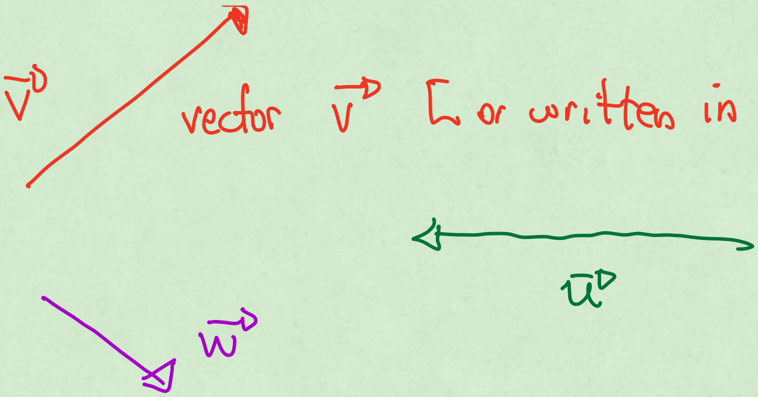


trajectory of
the particle gives you the original
parabola



arrow = (position) vector → gives you a direction in space
→ gives you a size or distance between the points.

vector $\vec{v}$ [or written in bold]

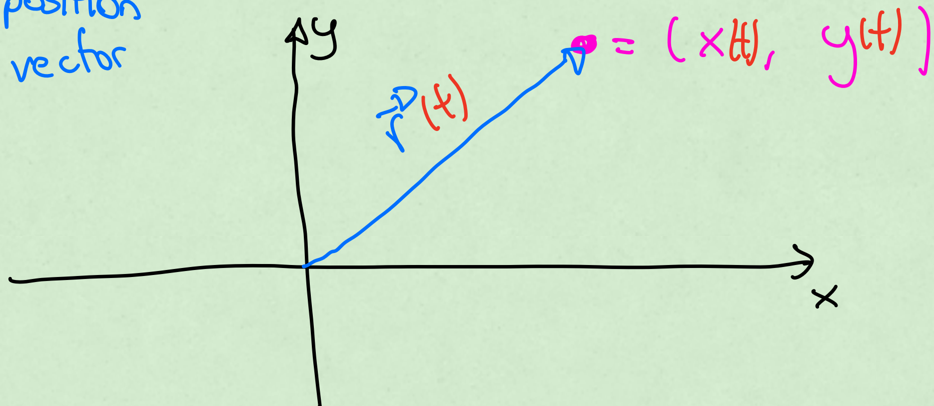


add/subtract vectors (arrows)

multiply vectors → dot product

→ cross product

$\vec{r}$ = position vector



$\vec{r}(t)$ = position of the particle at time t.

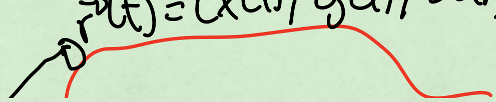
$\vec{r}(t) = (t, t^2)$ $\quad x(t) = t$

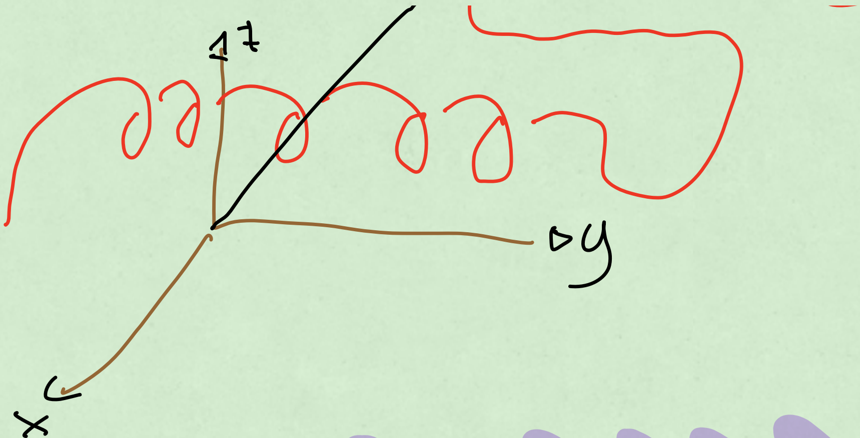
$y(t) = t^2$

$\frac{d\vec{r}}{dt} = (1, 2t)$ = velocity of the particle $\vec{v}(t)$

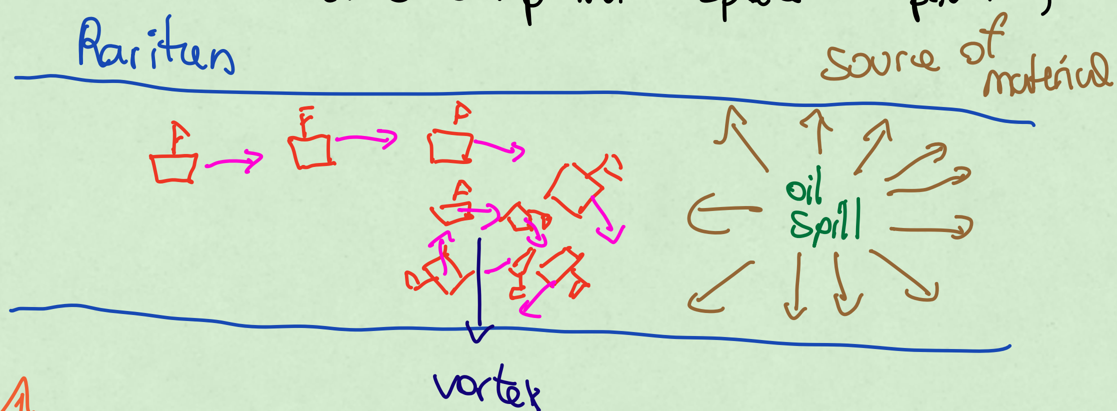
$\frac{d^2\vec{r}}{dt^2} = (0, 2)$ = acceleration $\vec{a}(t)$

$\vec{r}(t) = (x(t), y(t), z(t))$

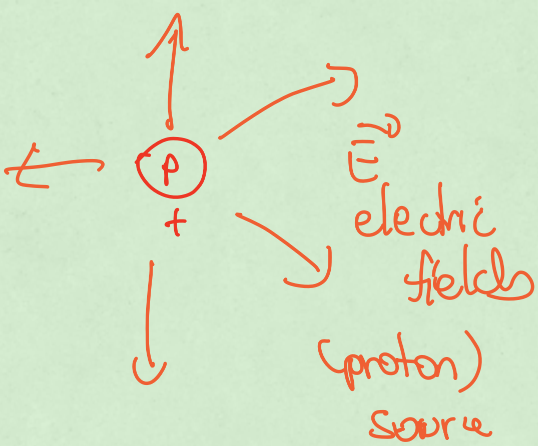


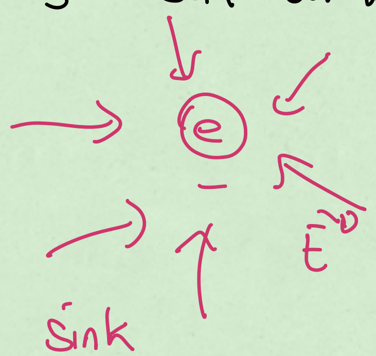


vector field = one arrow drawn at each point of space



(vorticity or curl not vanishing)





12.1 – 12.2

vectors:

$\vec{v}$

vector → direction

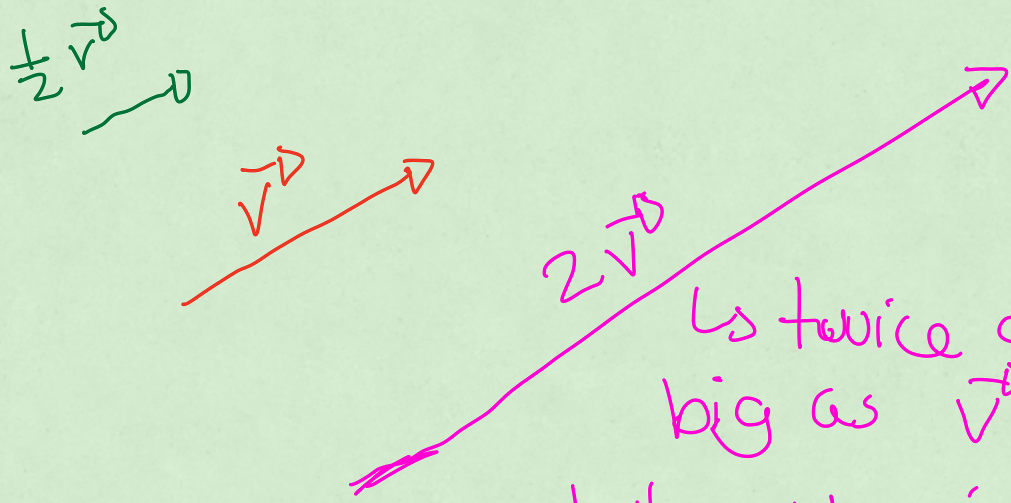
↳ size of a vector

symbol is $|\vec{v}|$ or $\|\vec{v}\|$ $\left\{\begin{array}{l} \text{size of a vector} \\ = \text{magnitude vector} \\ = \text{norm vector} \end{array}\right.$

$|\vec{v}|$ = size of a vector

$\underbrace{\phantom{|\vec{v}|}}$ = a non-negative number

behaves like an absolute value



↳ twice as big as $\vec{v}$, but points in the same direction

analogy

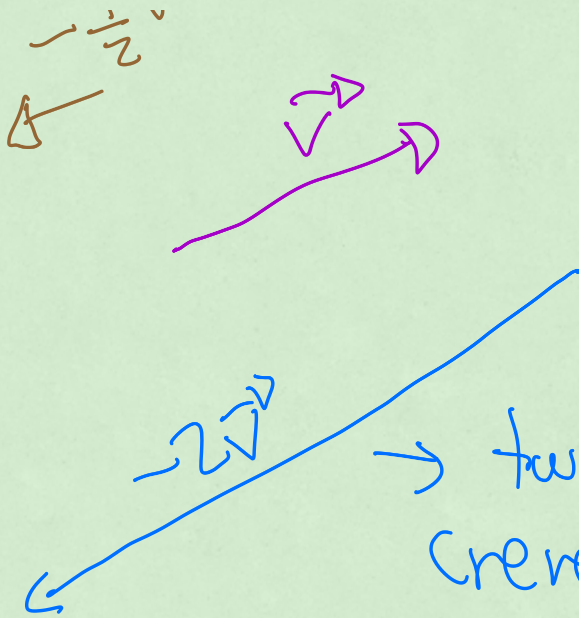
$$|2x| = 2|x|$$

$$|2\vec{v}| = 2|\vec{v}|$$

analogy

$$|-2x| = 2|x|$$

$|\vec{v}$



→ twice as big (reverse the direction)

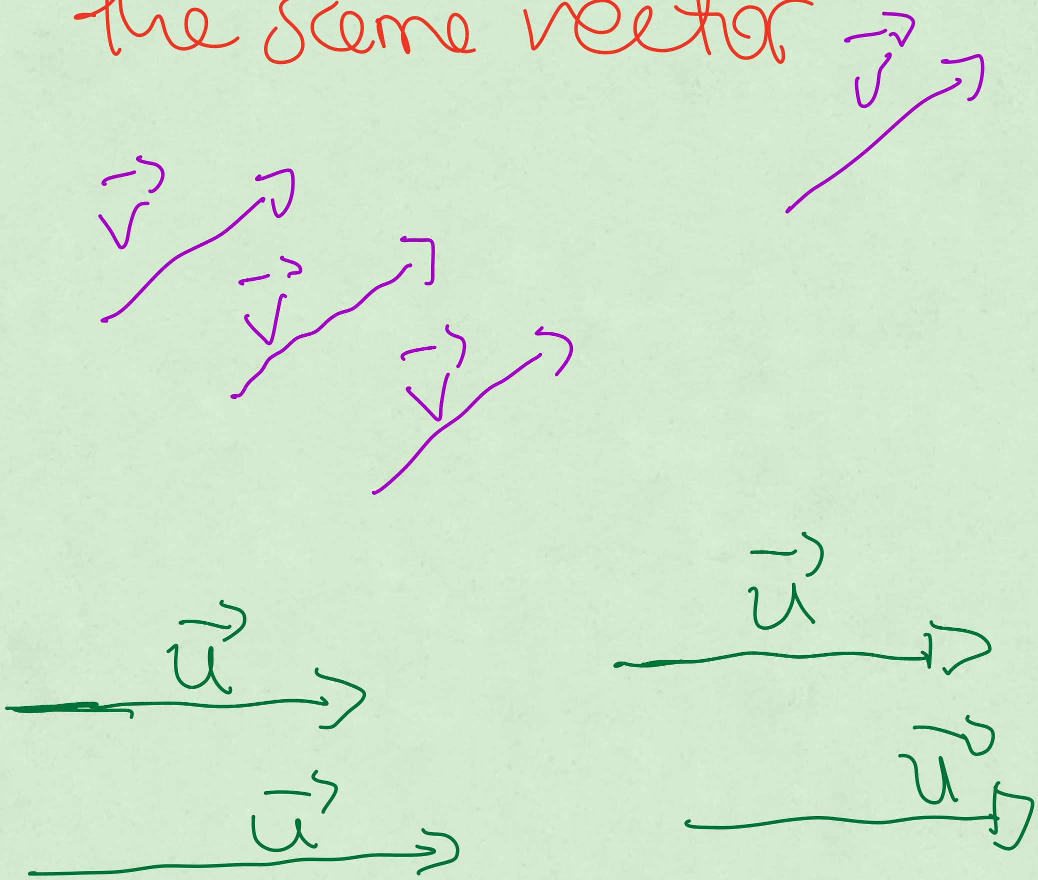
$$|-2\vec{v}| = 2|\vec{v}|$$

Remark!

this is how you multiply a vector by a number

ordinary terminology	fancy terminology
arrows	vectors
size	magnitude or norm
numbers (integers, π, e, $\sqrt{2}$)	scalars

if a vector is moved while preserving its size and direction then it is considered the same vector



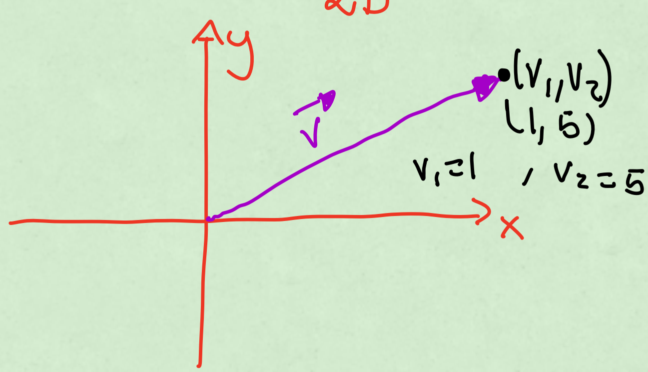
Lecture 2 (12.2-12.3)

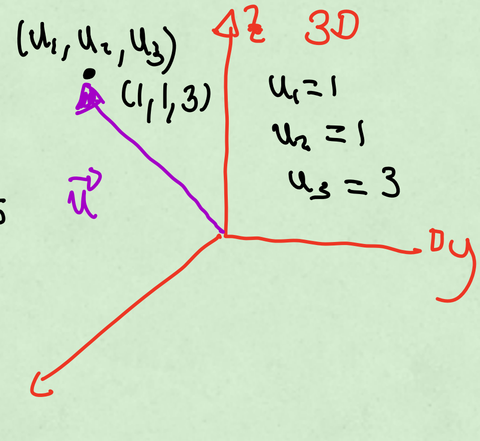
Vectors and some operations

↳ add/subtract vectors

↳ dot product (vector multiplication)

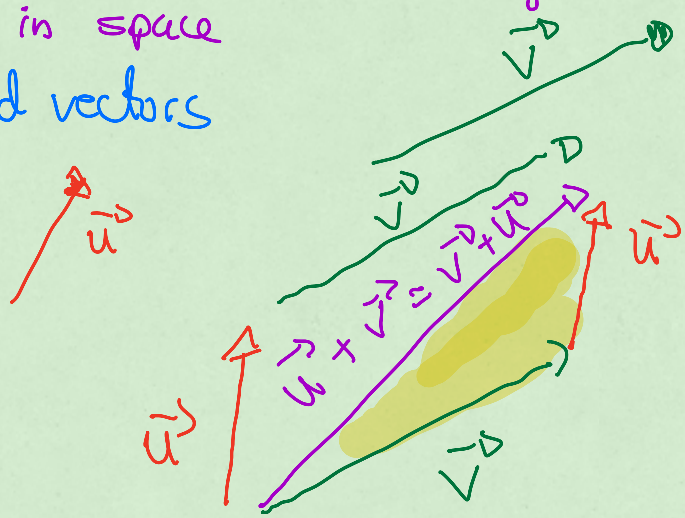
Drawing vectors





The vectors start at the origin - you can use them to specify a point in space

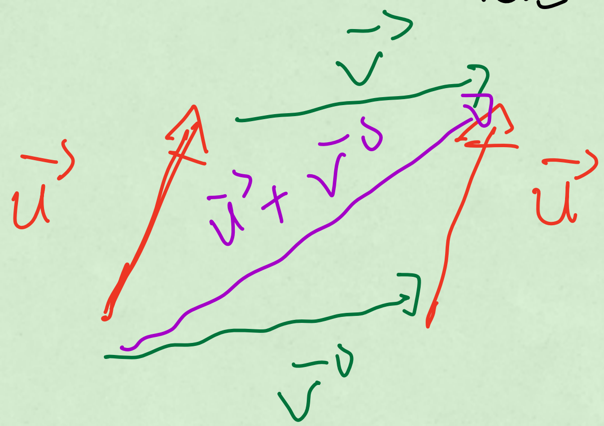
How to add vectors

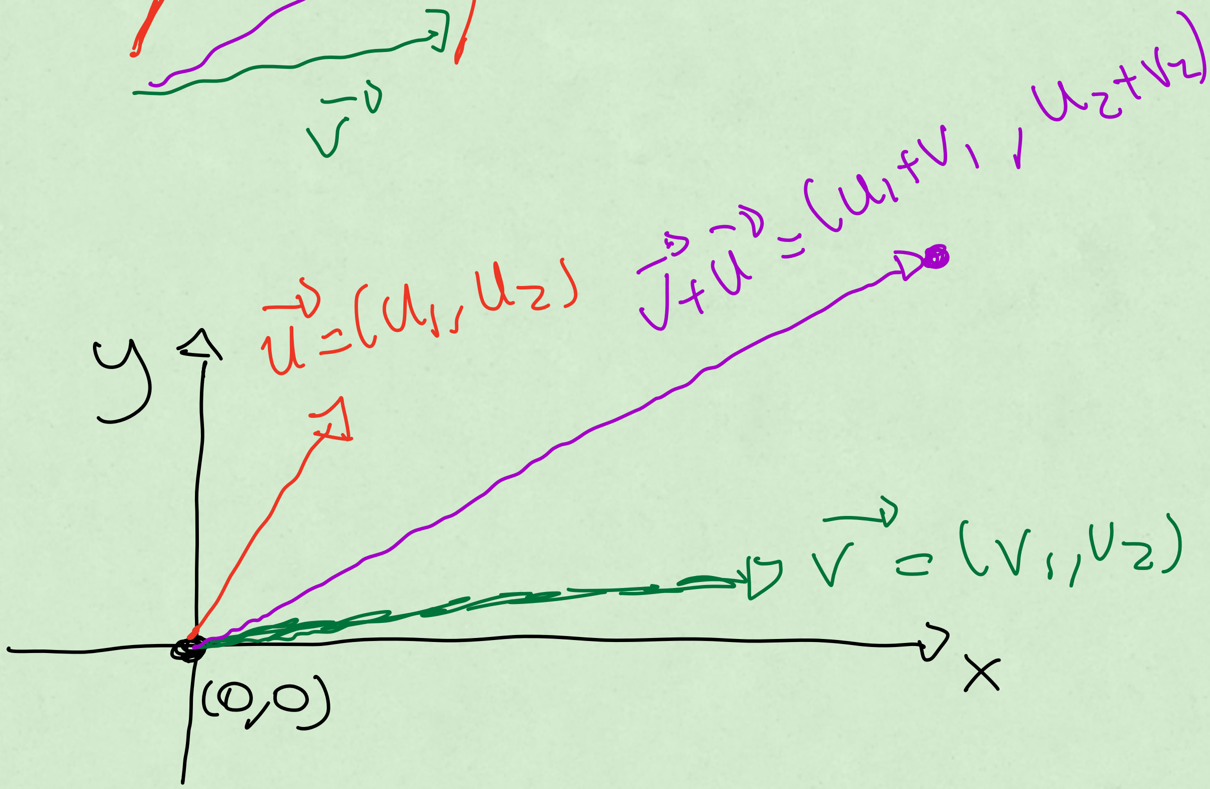


The way to add vectors:

make a parallelogram whose sides

is represented by the two vectors, one of the diagonals gives you the sum of the vectors





$\vec{u} = (1, 5)$

$\vec{v} = (3, 8)$

$\vec{u}+\vec{v} = (4, 13)$

$\vec{w} = (-1, 7)$

$\vec{u} + \vec{w} = (0, 12)$

$\vec{u} = (1, -1, 5)$

$\vec{v} = (0, 1, 7)$

$\vec{u} + \vec{v} = (1, 0, 12)$

$\vec{u} = (1, 2, 0, 5)$

$\vec{v} = (1, 1, -5, 0)$

$\vec{u}+\vec{v}=(2,3,-5,5)$

$\boxed{\text{Flatland}}$

not allowed:

$\vec{u}=(1,3)$

$\vec{v}=(2,1,0)$

$\vec{w}=(1,1,1,1)$

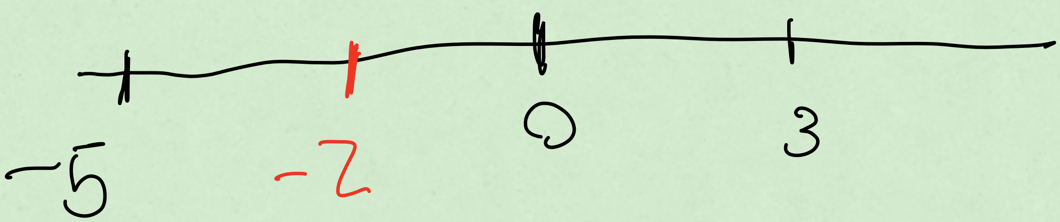
$\vec{u}+\vec{v}$ not defined!

so only add vectors with the same number of entries

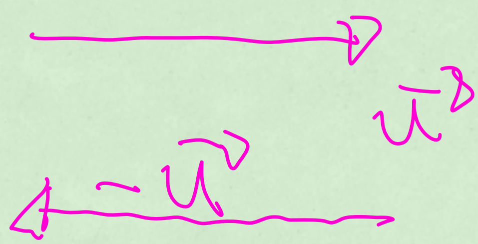
$\vec{v} + \vec{w}$ not defined

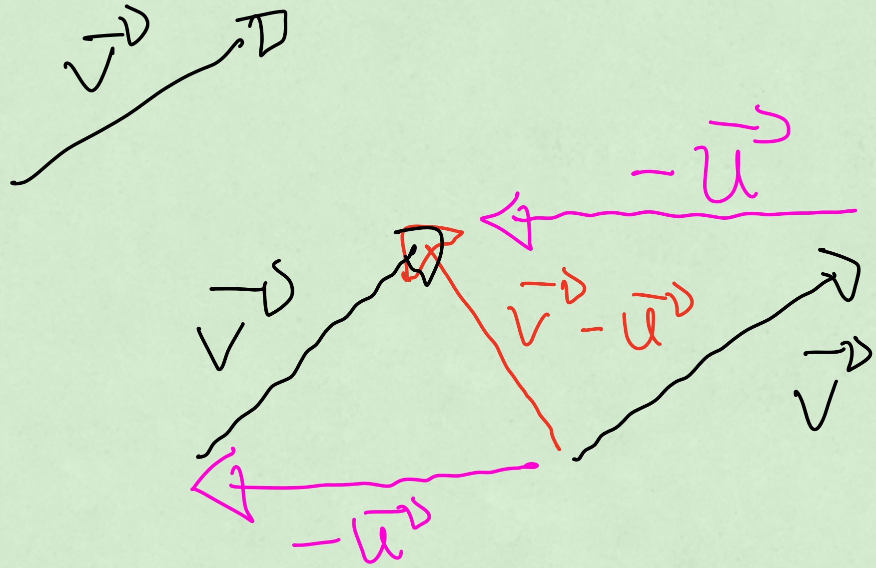
How to subtract vectors

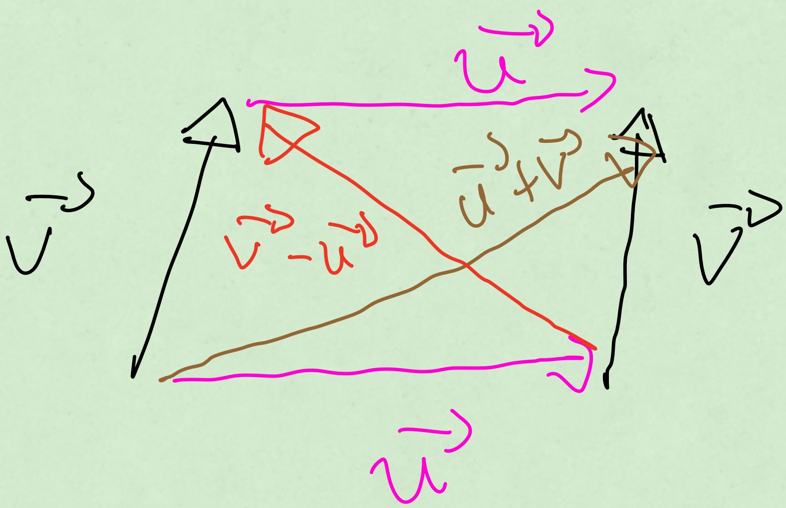
$$3 - 5 = 3 + (-5)$$



$$\vec{v} - \vec{u} = \vec{v} + (-\vec{u})$$







$\vec{v}-\vec{u}$: the other diagonal in the parallelogram

$\vec{u} = (1, 5)$

$$\vec{v} = (2, 7)$$

$$\vec{v} - \vec{u} = (1, 2)$$

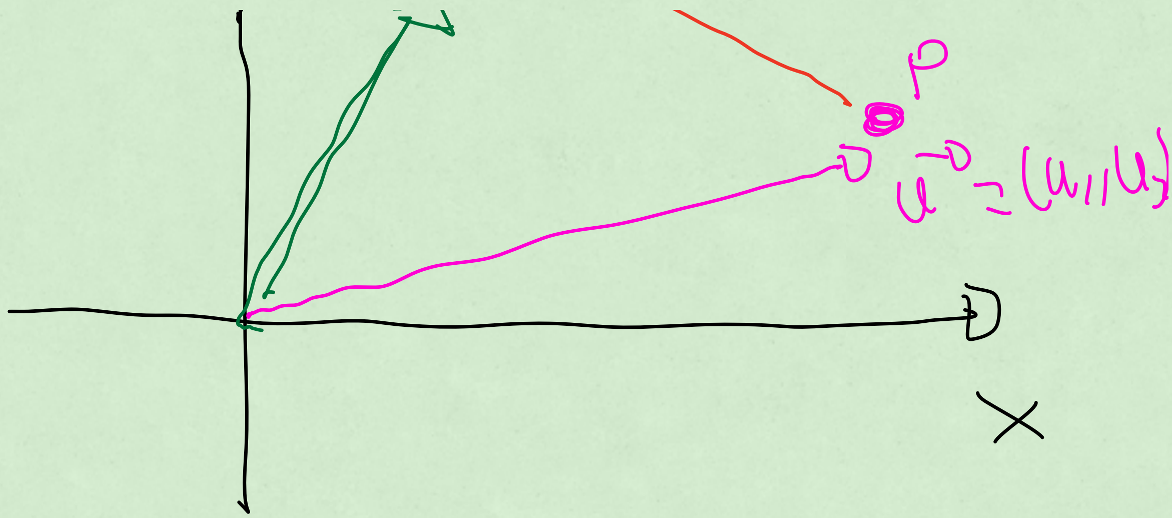
$$\vec{u} - \vec{v} = (-1, -2)$$

$$= -(\vec{v} - \vec{u})$$

$$\vec{u} - \vec{v} = -(\vec{v} - \vec{u})$$

$$\vec{v} = (v_1, v_2)$$

Δy $i\theta$ $\vec{v} - \vec{u}$



P is represented by a vector $(u_1, u_2) = \vec{u}$

Q is represented by a vector $(v_1, v_2) = \vec{v}$

$\vec{v} - \vec{u} = (v_1 - u_1, v_2 - u_2)$
is the vector that goes from point P

to point Q

$P = (x_1, y_1)$

$Q = (x_2, y_2)$

$\overrightarrow{PQ} = Q - P =$ vector from P to Q

$= (x_2 - x_1, y_2 - y_1)$

distance from P to Q

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

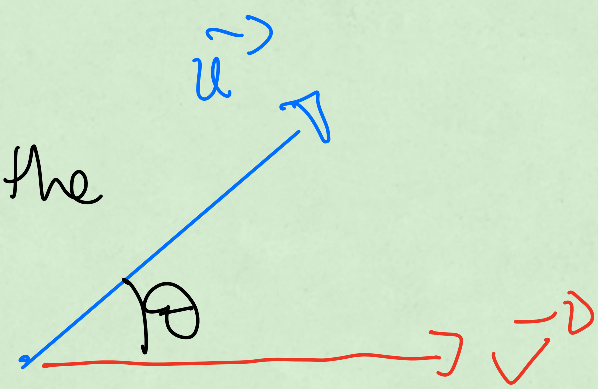
Dot product

(also known as scalar product or inner product)

$\vec{u} \cdot \vec{v}$ = this will be a number, <u>not</u> a new vector

θ = angle between the vector

↓

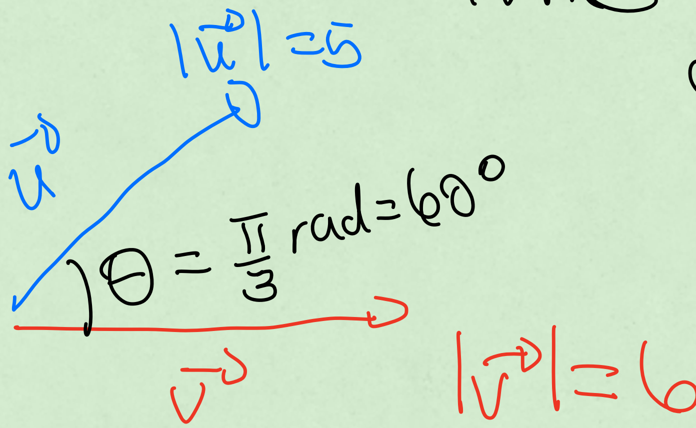


the one which is between $0°$ and $180°$ (or 0 and π rad)

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$$

Key Formula/Definition

= multiply the lengths of the vectors times cosine angle



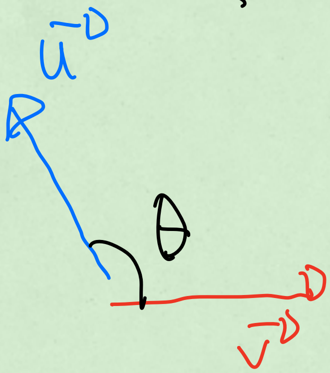
$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$$
$$= 5 \cdot 6 \cdot \cos\left(\frac{\pi}{3}\right)$$

$= 30 \cdot \frac{1}{2}$

$= \boxed{15}$

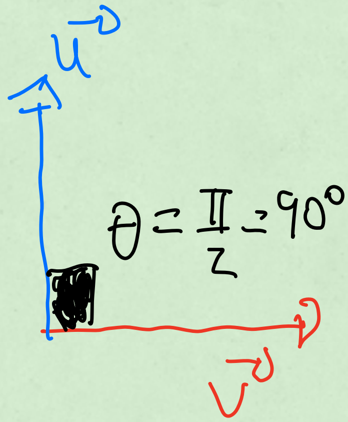
$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$

$90^\circ < \theta \leq 180^\circ$



$\underbrace{\vec{u} \cdot \vec{v} < 0}$

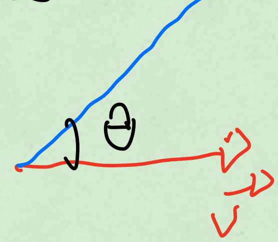
negative dot product



$\vec{u} \cdot \vec{v} = 0$

$0 \leq \theta < 90^\circ$

acute



$\underbrace{\vec{u} \cdot \vec{v} > 0}$

positive dot product

$$\vec{0} = (0,0)$$

$$\vec{0} = (0,0,0)$$

$$\boxed{\vec{u}\cdot\vec{0} = 0}$$

$$\boxed{\vec{u}+\vec{0} = \vec{u}}$$

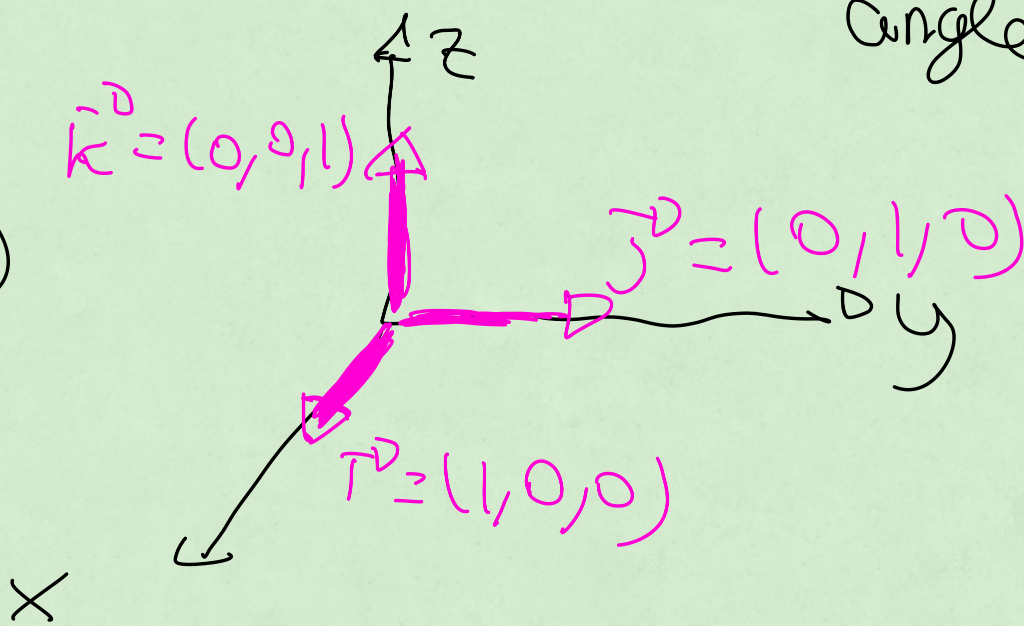
Properties of dot product

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

2. $\vec{u}\cdot\vec{v}$
   - positive = acute angle
   - zero = perpendicular (orthogonal vectors)

↳ negative = obtuse angle

3



$$|\vec{i}| = |\vec{j}| = |\vec{k}| = \underline{1}$$

$$\vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = 0$$

↓

mutually perpendicular

next time

$$\vec{u} = (1, 0, 3)$$

$$\vec{v} = (-1, 1, 2)$$

$$\vec{u} = \vec{i} + 3\vec{k}$$

$$\vec{v} = -\vec{i} + \vec{j} + 2\vec{k}$$

$$\vec{u} \cdot \vec{v}$$

$$= (\vec{i} + 3\vec{k}) \cdot (-\vec{i} + \vec{j} + 2\vec{k})$$

$$= \vec{i} \cdot (-\vec{i}) + (3\vec{k}) \cdot (2\vec{k})$$

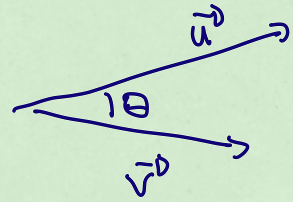
$$= -1 + 6$$

$$= 5$$

$$= 1 \cdot (-1) + (0 \cdot 1) + (3 \cdot 2)$$

Lecture 3 (12.3 - 12.4)

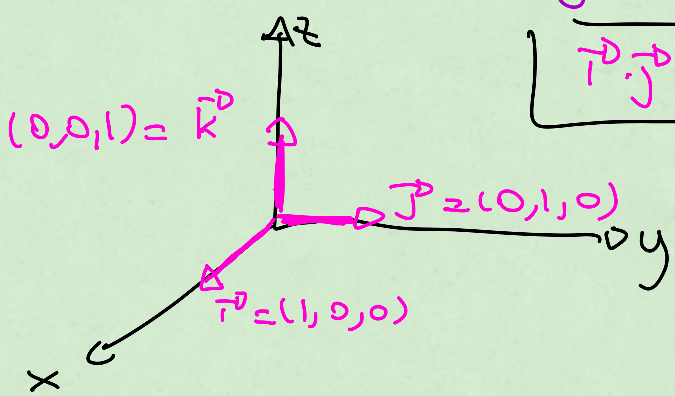
Last time



$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$$

Properties

$\vec{u} \cdot \vec{v} = 0$, vectors are perpendicular (orthogonal) to one another



$$\boxed{\vec{i} \cdot \vec{j} = \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0}$$

$$\vec{i} \cdot \vec{i} = |\vec{i}|\,|\vec{i}| \cos\theta = 1 \cdot 1 \cdot \cos 0 = 1$$

$\longrightarrow \vec{i}$

$\longrightarrow \vec{i}$

$$\boxed{\vec{i} \cdot \vec{i} = 1}$$

$$\boxed{\vec{j} \cdot \vec{j} = 1} \qquad \boxed{\vec{k} \cdot \vec{k} = 1}$$

How to find dot product

$$\vec{u} = (2,3,5) = 2\vec{i} + 3\vec{j} + 5\vec{k}$$

$$\vec{v} = (2,4,6) = 2\vec{i} + 4\vec{j} + 6\vec{k}$$

$$\vec{u}\cdot\vec{v} = (2\vec{i}+3\vec{j}+5\vec{k})\cdot(2\vec{i}+4\vec{j}+6\vec{k})$$

$$= (2\cdot 2)\overset{1}{\vec{i}\cdot\vec{i}} + (2\cdot 4)\underset{0}{\vec{i}\cdot\vec{j}} + (2\cdot 6)\underset{0}{\vec{i}\cdot\vec{k}}$$

$$+ (3\cdot 2)\overset{0}{\vec{j}\cdot\vec{i}} + (3\cdot 4)\overset{1}{\vec{j}\cdot\vec{j}} + (3\cdot 6)\overset{0}{\vec{j}\cdot\vec{k}}$$

$$+ (5\cdot 2)\underset{0}{\vec{k}\cdot\vec{i}} + (5\cdot 4)\underset{0}{\vec{k}\cdot\vec{j}} + (5\cdot 6)\underset{1}{\vec{k}\cdot\vec{k}}$$

$$= 2\cdot 2 + 3\cdot 4 + 5\cdot 6$$

$$= 4 + 12 + 30$$

$$= 46$$

$$\vec{u} = (2, 3, 5)$$

$$\vec{v} = (2, 4, 6)$$

$$\vec{u}\cdot\vec{v} = 2\cdot 2 + 3\cdot 4 + 5\cdot 6$$

$$= 46$$

General Formula

$$\vec{u} = (u_1, u_2, u_3)$$

$$\vec{v} = (v_1, v_2, v_3)$$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$= \underline{u_1 v_1 + u_2 v_2 + u_3 v_3}$$

tends to be more convenient.

Property of the dot product

$$\vec{v} = (v_1, v_2, v_3)$$

$$\vec{v} \cdot \vec{v} = |\vec{v}||\vec{v}|\cos\theta$$
$$= |\vec{v}|^2 \cos 0$$
$$= |\vec{v}|^2$$

$$\vec{v} \cdot \vec{v} = (v_1, v_2, v_3) \cdot (v_1, v_2, v_3)$$

$$= v_1^2 + v_2^2 + v_3^2$$

$$|\vec{v}|^2 = v_1^2 + v_2^2 + v_3^2$$

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

3D Pythagoras

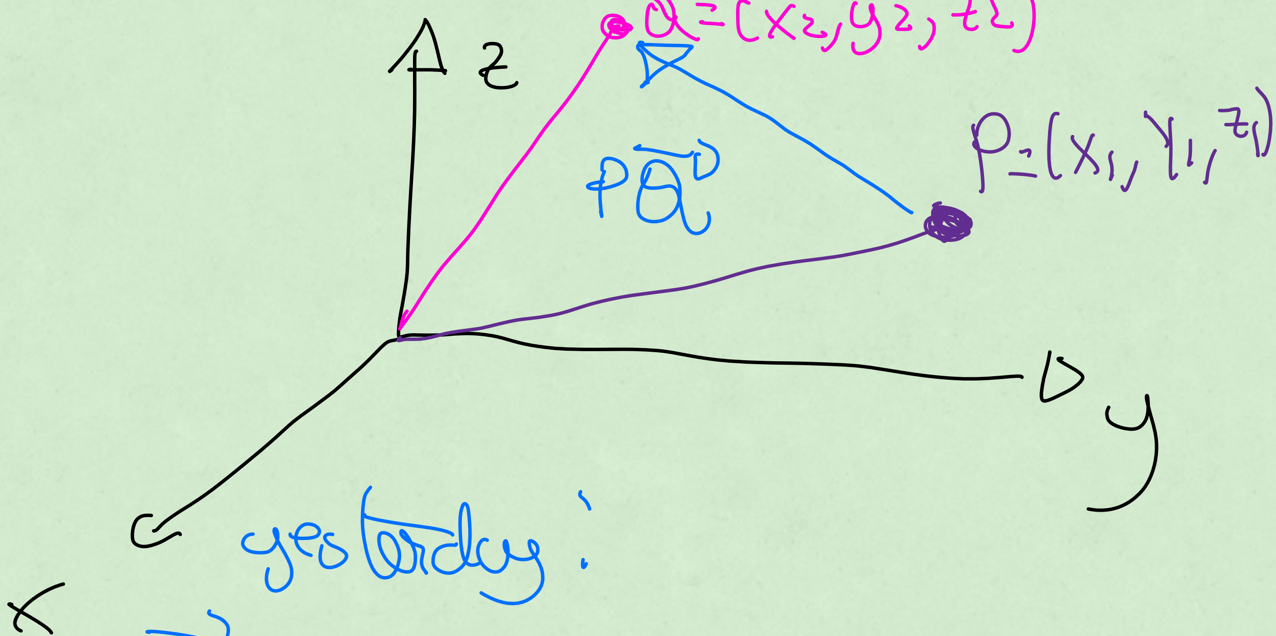
examples

$$\vec{v} = (1, -2, 5)$$

$$|\vec{v}| = \sqrt{1^2 + (-2)^2 + 5^2}$$

$$= \sqrt{1 + 4 + 25}$$

$$= \sqrt{30}$$



yesterday:

$$\overrightarrow{PQ} = Q - P = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

distance P to Q:

$$= |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Angle between two

- vectors

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

$$\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \cos\theta$$

$$\boxed{\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)}$$

example:

angle between

$\vec{v} = (-1, 2, 5)$

$\vec{u} = (1, 1, 1)$

$$\theta = \arccos\left(\frac{(1,1,1)\cdot(-1,2,5)}{\|(1,1,1)\|\,\|(-1,2,5)\|}\right)$$

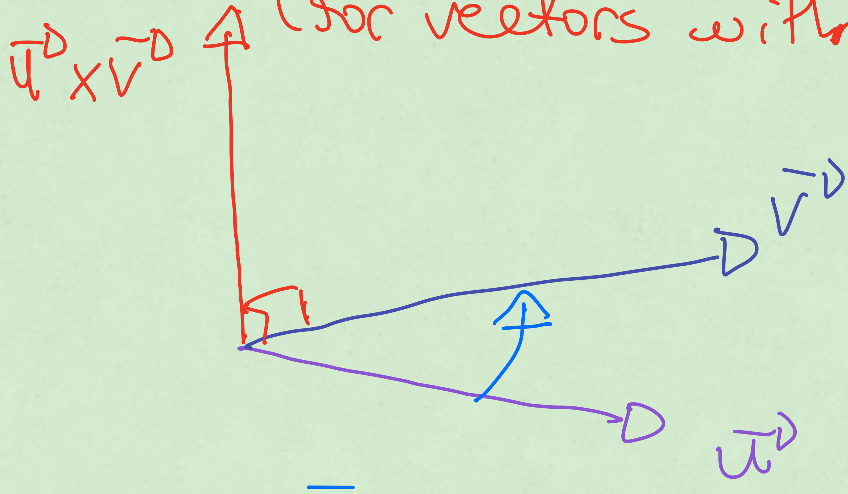
$$= \arccos\left(\frac{-1+2+5}{\sqrt{1^2+1^2+1^2}\,\sqrt{(-1)^2+2^2+5^2}}\right)$$

$$= \arccos\left(\frac{6}{\sqrt{3}\,\sqrt{30}}\right)$$

$$= \arccos\left(\frac{6}{\sqrt{90}}\right)$$

(12.4) Cross Product

(for vectors with 3 entries)



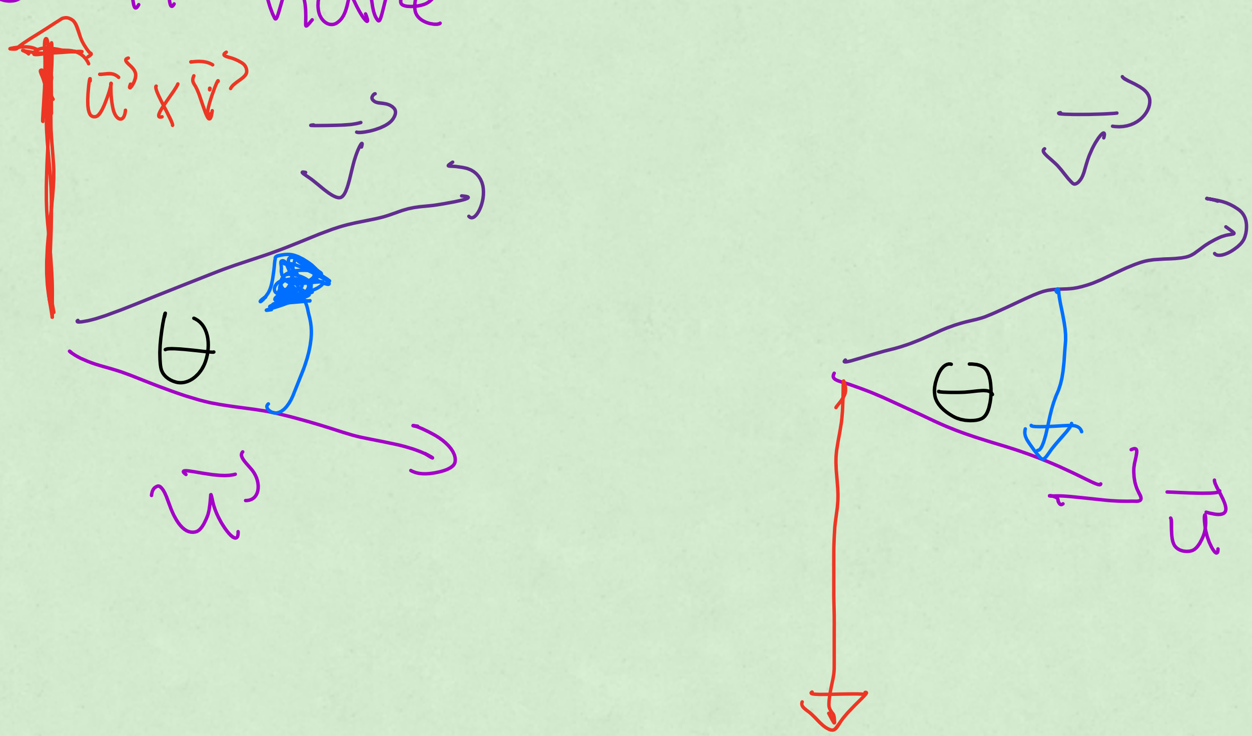
$\vec{u} \times \vec{v}$ = cross product

= new vector

= vector perpendicular to both $\vec{u}$ and $\vec{v}$

Right hand rule:

decides the direction that the cross product will have

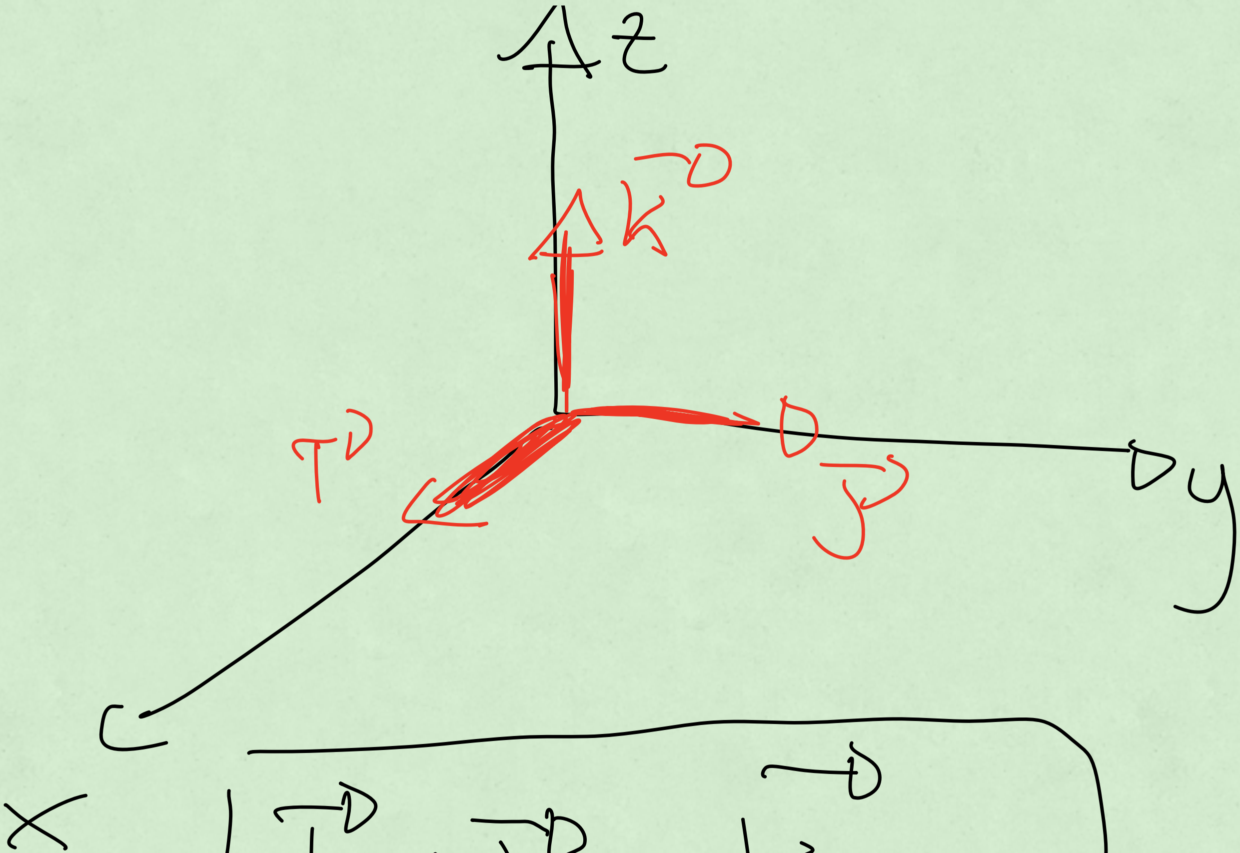


$\vec{v} \times \vec{u}$

$$\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$$

the order matters!

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$$



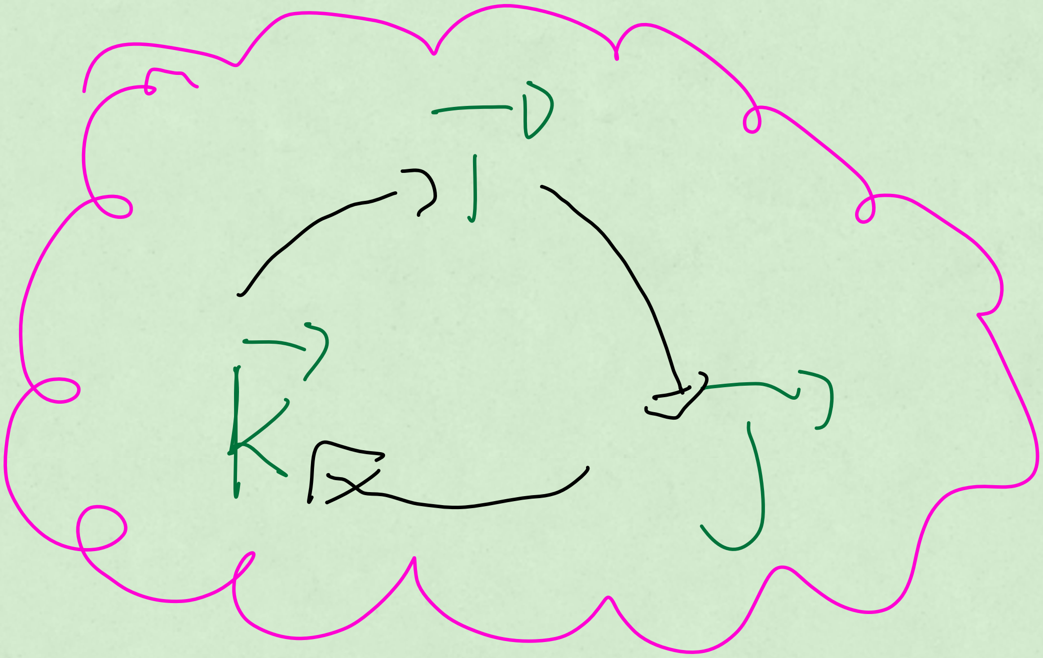
$$\vec{i} \times \vec{j} = \vec{k}$$

$$\vec{j} \times \vec{i} = -\vec{k}$$

$$\vec{j} \times \vec{k} = \vec{i}$$

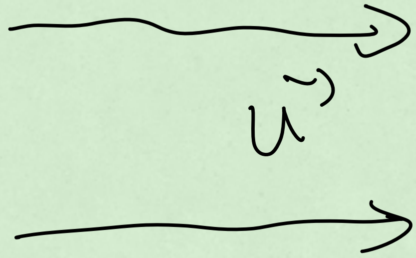
$$\vec{k} \times \vec{j} = -\vec{i}$$

$$\vec{k} \times \vec{i} = \vec{j}$$
$$\vec{i} \times \vec{k} = -\vec{j}$$



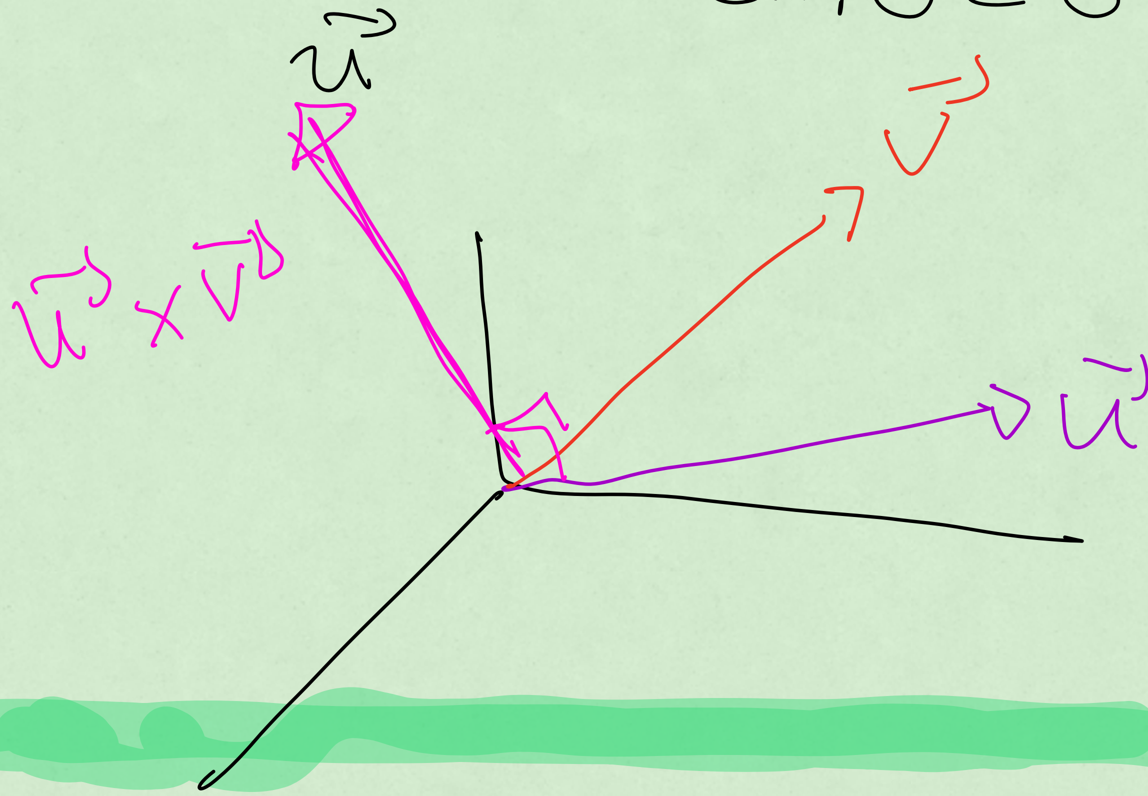
Properties of cross product

$$\vec{u} \times \vec{u} = \vec{0} = (0,0,0)$$



$\theta = 0$

$\sin\theta = 0$

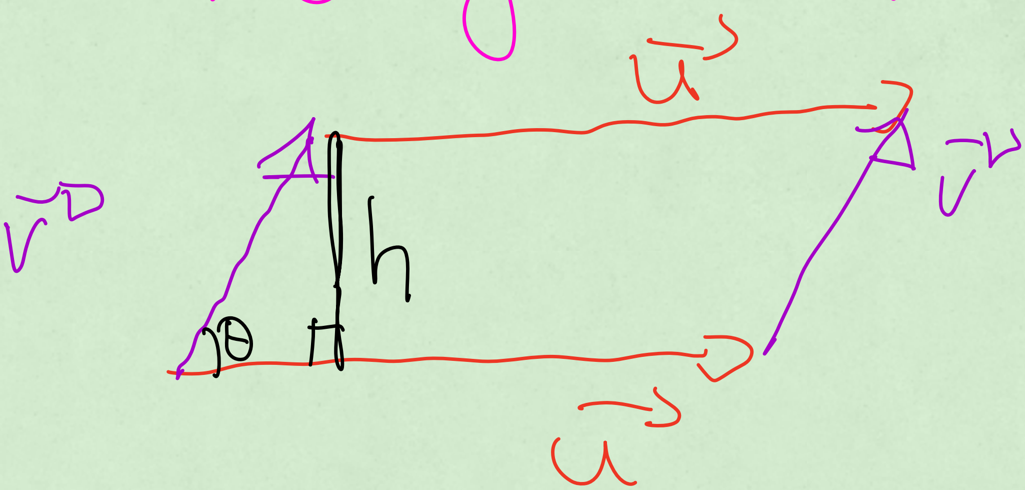


if $\vec{u}, \vec{v}$ are parallel
(or antiparallel)
(angle $\theta = 0^\circ$ or 180°)

then

$$\vec{u} \times \vec{v} = \vec{v} \times \vec{u} = \vec{0}$$

Parallelogram:



base $= |\vec{u}|$

height $= |\vec{v}| \sin\theta$

$$\sin\theta = \frac{h}{|\vec{v}|}$$

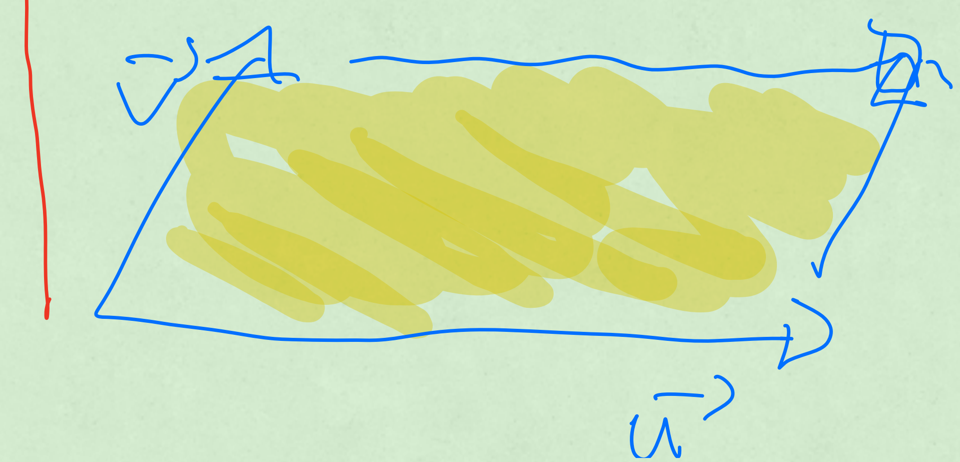
area

$= \text{base} \cdot \text{height}$

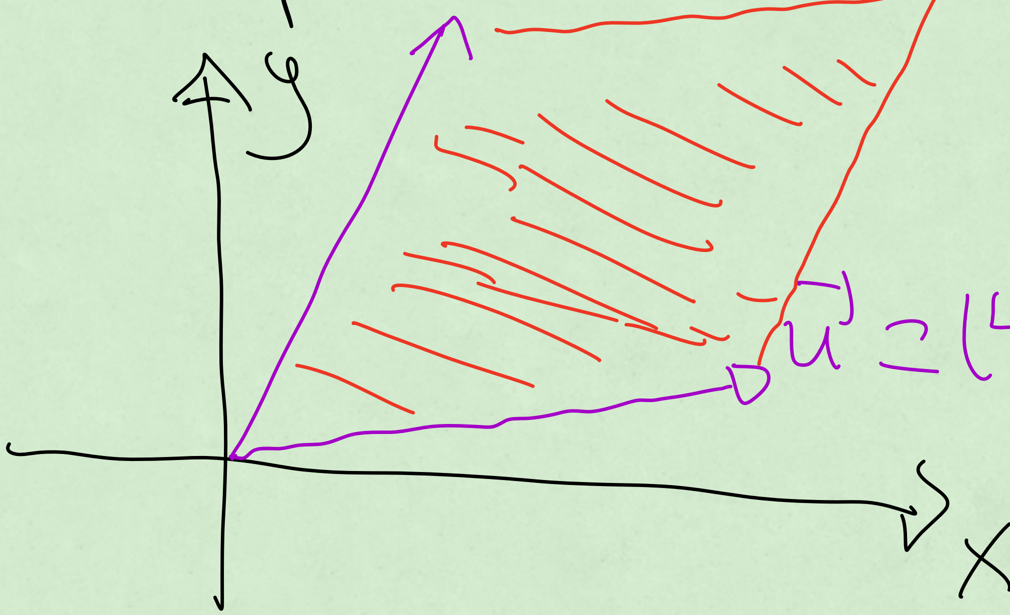
$= |\vec{u}|\,|\vec{v}|\sin\theta$

$= |\vec{u} \times \vec{v}|$

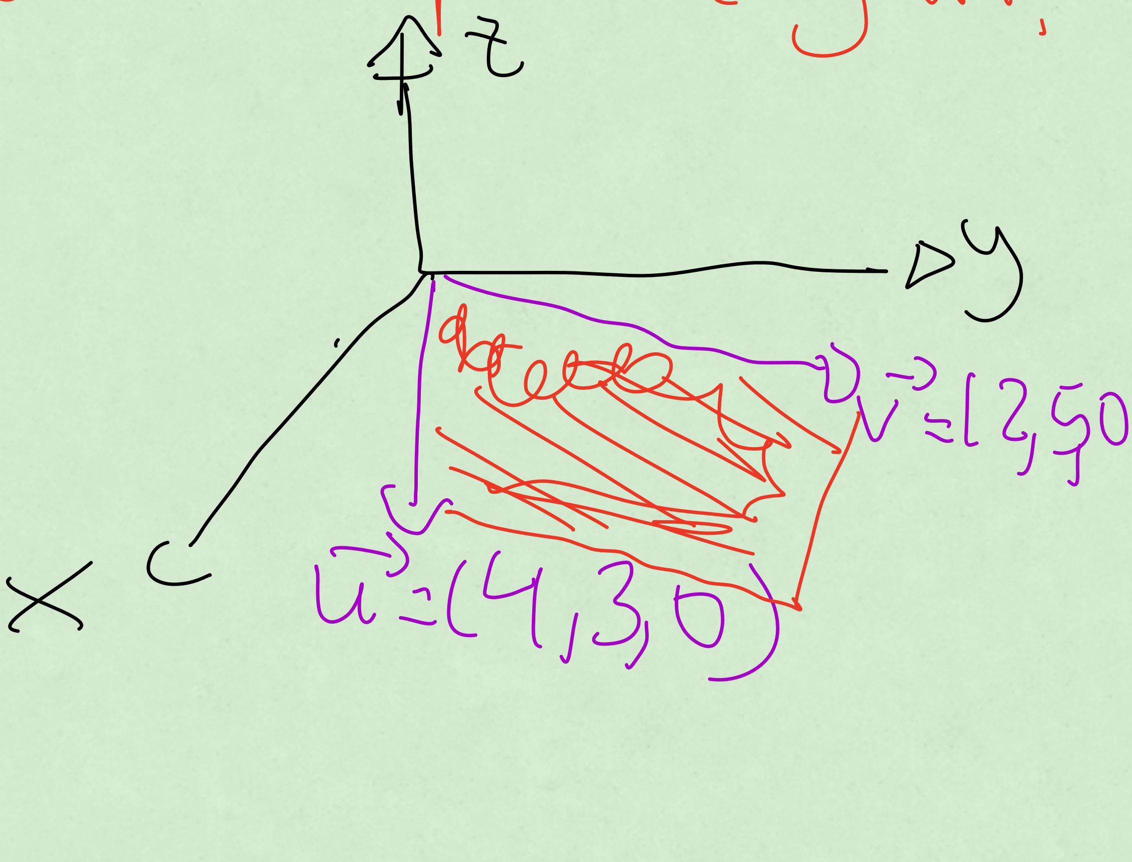
$\vec{u} \times \vec{v}$ = vector whose size is the area of the parallelogram



example: $\vec{v} = (2,5)$



area of parallelogram:



$$\vec{u} \times \vec{v}$$

$$= (4\vec{i} + 3\vec{j} + 0\vec{k})$$

$$\times (2\vec{i} + 5\vec{j} + 0\vec{k})$$

$$= (4\vec{i} + 3\vec{j}) \times (2\vec{i} + 5\vec{j})$$

$$= (4 \cdot 2)\underbrace{\vec{i} \times \vec{i}}_{\vec{0}} + (4 \cdot 5)\underbrace{\vec{i} \times \vec{j}}_{\vec{k}}$$

$$+ (3 \cdot 2)\underbrace{\vec{j} \times \vec{i}}_{-\vec{k}} + (3 \cdot 5)\underbrace{\vec{j} \times \vec{j}}_{\vec{0}}$$

$$= 20\vec{k} - 6\vec{k}$$

$$= 14\vec{k}$$

$$= (4 \cdot 5 - 3 \cdot 2)\vec{k}$$

$$\vec{u} = (4, 3)$$

$$\vec{v} = (2, 5)$$

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$$

$\det A = 4 \cdot 5 - 2 \cdot 3$

Formula for a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det A = ad - bc$$

$$\det \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} = 1 \cdot 5 - 2 \cdot 4$$
$$= 5 - 8$$

$$= -3$$

$$\det\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$$

$$= 2 \cdot 5 - (-1) \cdot 3$$

$$= 10 + 3$$

$$= 13$$

Lecture 4 (12.4-12.5)

Determinants of matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad 2 \times 2 \text{ matrix}$$

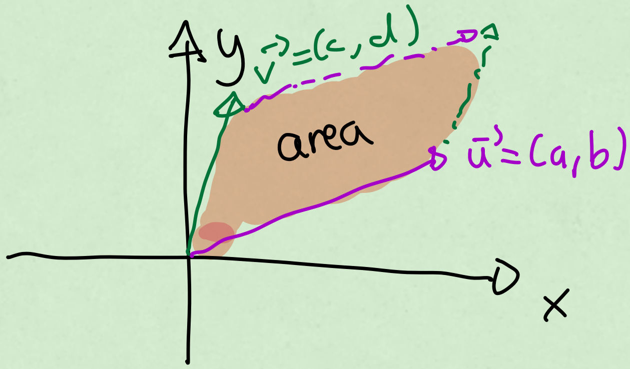
$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 7 \end{pmatrix}$$

$$\det A = 2 \cdot 7 - 1 \cdot 5 = 14 - 5 = 9$$

$$A = \begin{pmatrix} 2 & -4 \\ 5 & 6 \end{pmatrix}$$

$$\det A = 2 \cdot 6 - (-4) \cdot 5 = 12 + 20 = 32$$



$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{area} = |\vec{u} \times \vec{v}| = \text{absolute value of } \det A$$
$$= |ad - bc|$$

3x3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

volume

$V = |\det A|$

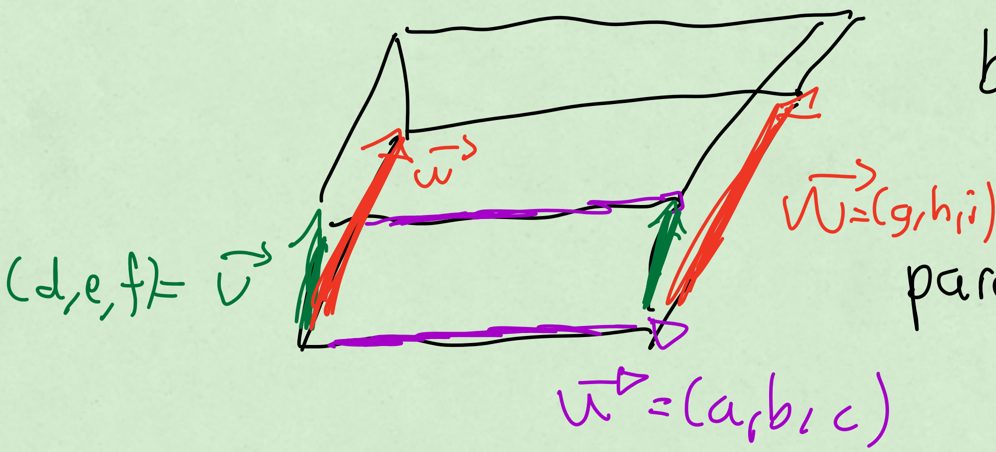
= absolute value of determinant

$\det A$ is a number

tilted box

or

parallelepiped



$\int f(x)\,dx$ calc 1

$\iint f(x,y)\,dy\,dx$ in this course

$\iint f(r,\theta)\,r\,dr\,d\theta$

↳ this comes from a determinant

Method for finding determinant

$$\begin{array}{ccc} + & - & + \\ \end{array}$$
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= +a \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

---

example:

$$\begin{array}{ccc} + & - & + \\ \end{array}$$
$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 7 & 2 & 2 \end{pmatrix}$$

$\det A$

$$= +1 \det\begin{pmatrix} 4 & 5 \\ 2 & 2 \end{pmatrix} - 2 \det\begin{pmatrix} -1 & 5 \\ 7 & 2 \end{pmatrix} + 3 \det\begin{pmatrix} -1 & 4 \\ 7 & 2 \end{pmatrix}$$

$$= (4\cdot 2 - 5\cdot 2) - 2(-2-35) + 3(-2-28)$$

$$= -2 - 2(-37) + 3(-30)$$

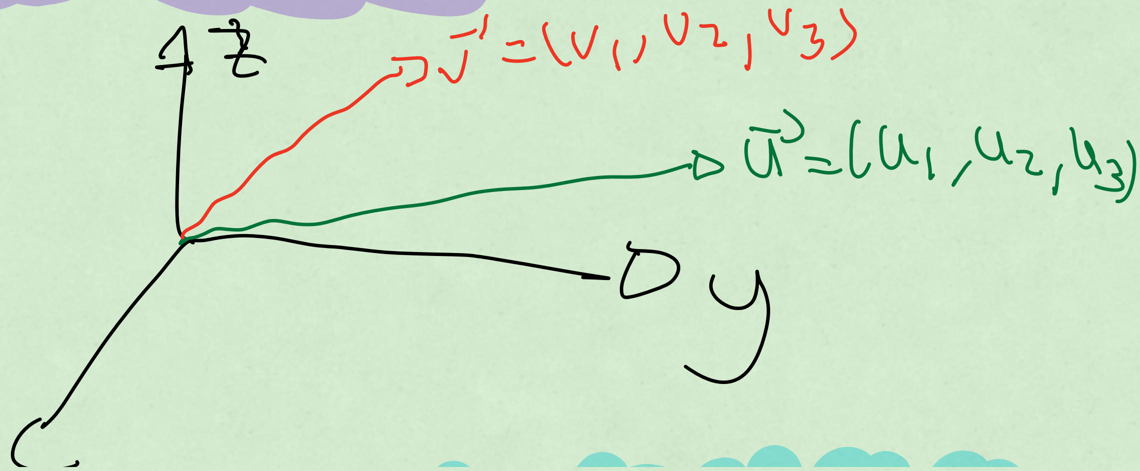
$$= -2 + 74 - 90$$

$$= 72 - 90$$

$$= -18$$

$$|\det A| = |-18| = 18$$

= volume of the parallelepiped obtained by taking the rows as vectors making of the box



$\times$

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

example:

$$\vec{u} = (1, 3, -5)$$
$$\vec{v} = (1, 7, 2)$$

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -5 \\ 1 & 7 & 2 \end{pmatrix}$$

$$= +\vec{i} \det \begin{pmatrix} 3 & -5 \\ 7 & 2 \end{pmatrix} - \vec{j} \det \begin{pmatrix} 1 & -5 \\ 1 & 2 \end{pmatrix} + \vec{k} \det \begin{pmatrix} 1 & 3 \\ 1 & 7 \end{pmatrix}$$

$$= \vec{i}(6+35) - \vec{j}(2+5) + \vec{k}(7-3)$$

$$= 41\vec{i} - 7\vec{j} + 4\vec{k}$$

$$= \langle 41, -7, 4 \rangle$$

$$(\vec{u} \times \vec{v}) \times \vec{u}$$

$$\times \vec{v}$$

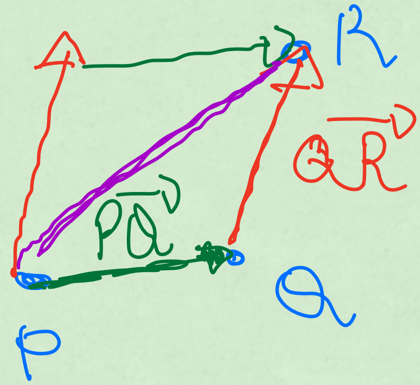
$$\times (\vec{u} \times \vec{v})$$

$$(\vec{u} \times \vec{v}) \times (\vec{u} \times \vec{v}) = \vec{0}$$

vector identities

(wikipedia)

area of a triangle



area of triangle with vertices P, Q, R

$= \frac{1}{2}$ area parallelogram with sides $\vec{PQ}$, $\vec{QR}$

$$= \frac{1}{2} \left| \vec{PQ} \times \vec{QR} \right|$$

Back to 12.3

vector projections

shadow vector

= projection of $\vec{u}$ onto $\vec{v}$

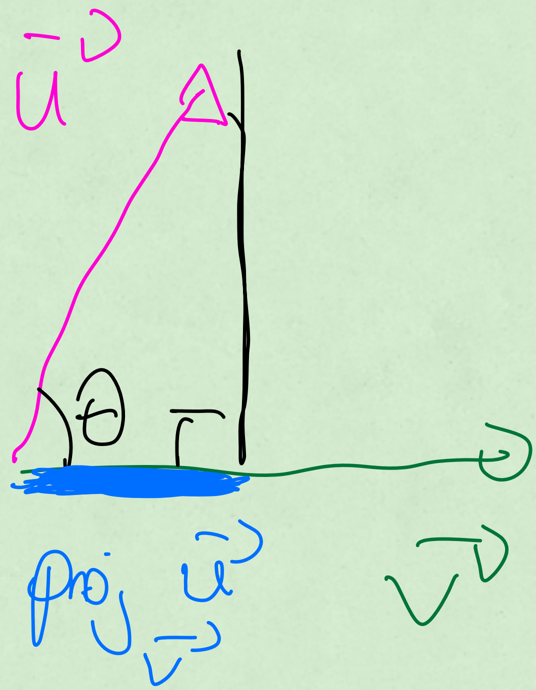
= $\text{proj}_{\vec{v}}\,\vec{u}$

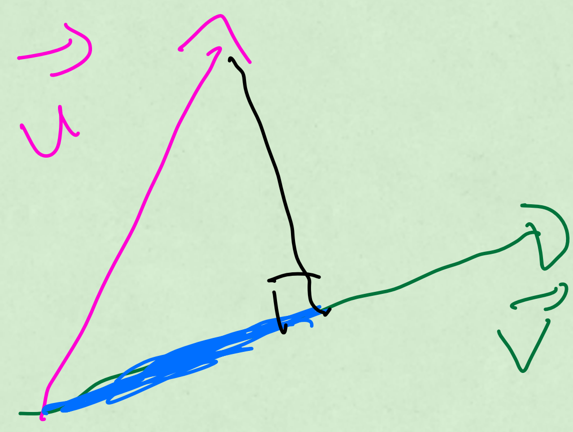
size of $\text{proj}_{\vec{v}}\,\vec{u}$

is called the

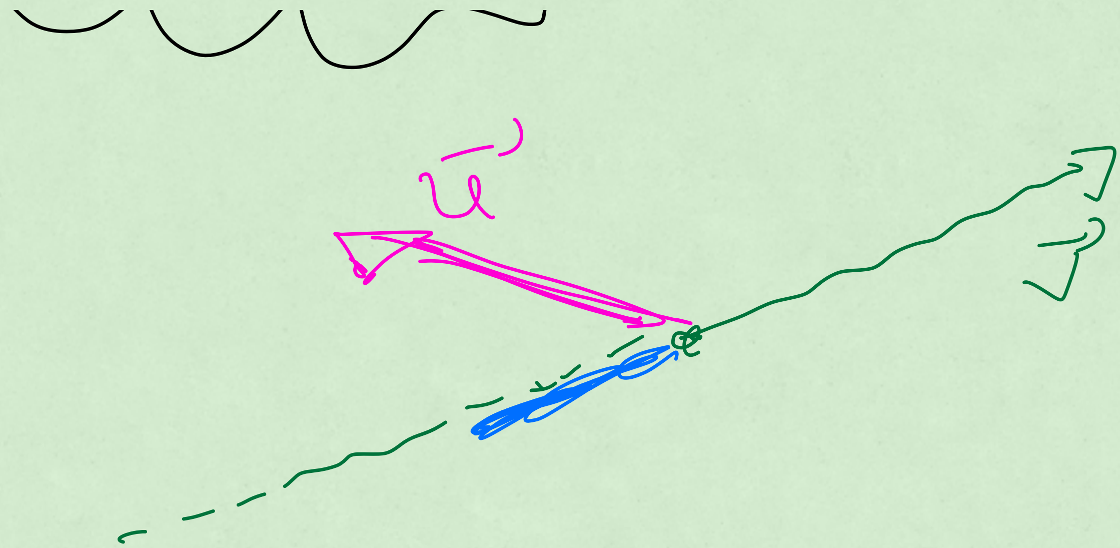
Scalar component of $\vec{u}$ in the direction of $\vec{v}$

$$\cos\theta = \frac{|\text{proj}_{\vec{v}}\vec{u}|}{|\vec{u}|}$$





$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

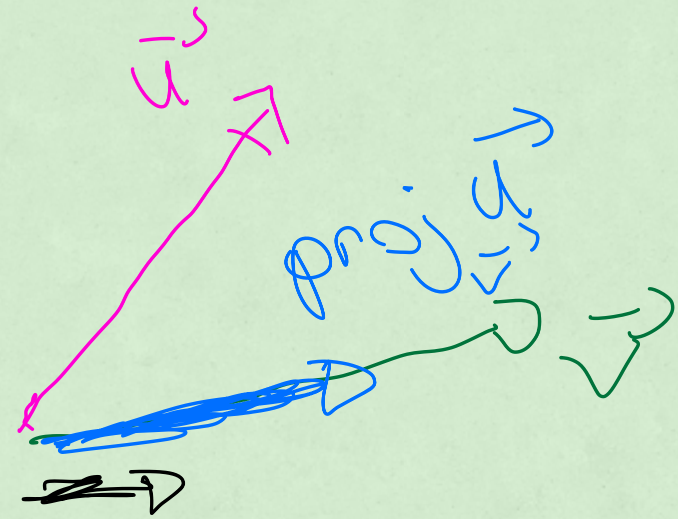


$$|\mathrm{proj}_{\vec{v}}\vec{u}| = |\vec{u}|\cos\theta$$

$$= |\vec{u}|\,\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

$$\boxed{|\mathrm{proj}_{\vec{v}}\vec{u}| = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}}$$

length of shadow



$$\frac{\vec{v}}{|\vec{v}|}$$

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{\operatorname{proj}_{\vec{v}} \vec{u} = \frac{(\vec{u} \cdot \vec{v})}{|\vec{v}|^2} \vec{v}}$$

example:

$$\vec{u} = (2, -1, 3)$$

$$\vec{v} = (1, 5, 7)$$

$$\operatorname{proj}_{\vec{v}} \vec{u}$$

$$= \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$

$$= \frac{(2,-1,3)\cdot(1,5,7)}{1^2+5^2+7^2}\vec{v}$$

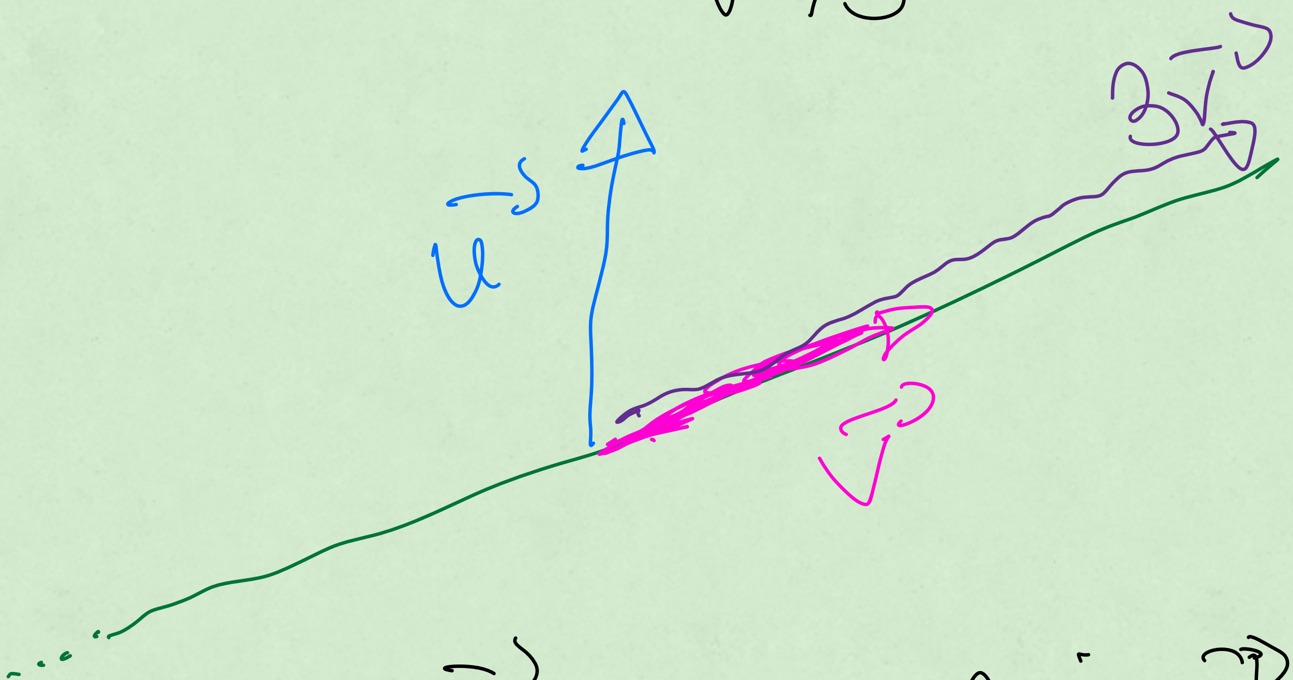
$$= \frac{2-5+21}{26+49}\vec{v}$$

$$= \frac{18}{75}\vec{v}$$

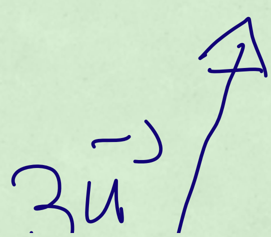
$$= \frac{18}{75}(1,5,7)$$

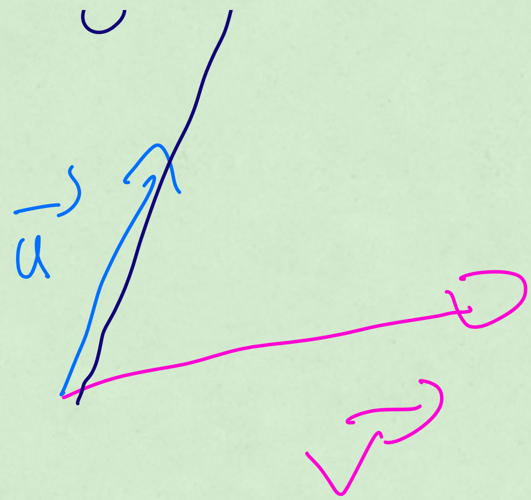
$$|\operatorname{proj}_{\vec{v}} \vec{u}| = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

$$= \frac{18}{\sqrt{75}}$$



$$\operatorname{proj}_{\vec{v}} \vec{u} = \operatorname{proj}_{3\vec{v}} \vec{u}$$



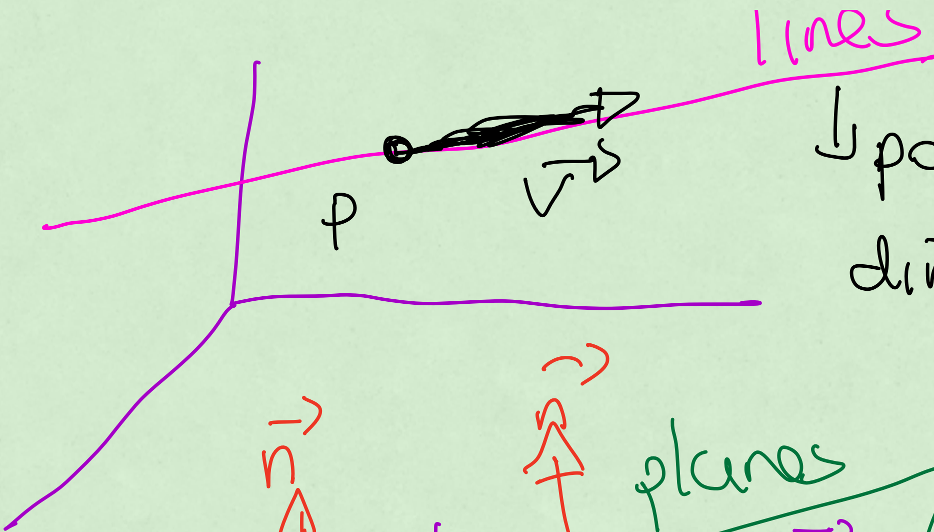


$$\mathrm{proj}_{\vec{v}}\, 3\vec{u} = 3\,\mathrm{proj}_{\vec{v}}\, \vec{u}$$

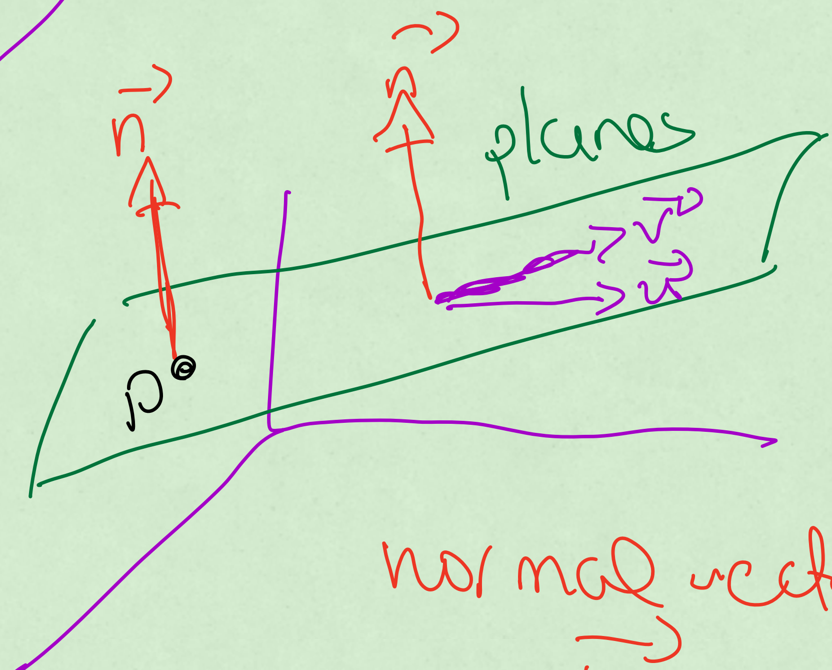
next time (12.5)

Big topic

planes and lines



↓ point + direction

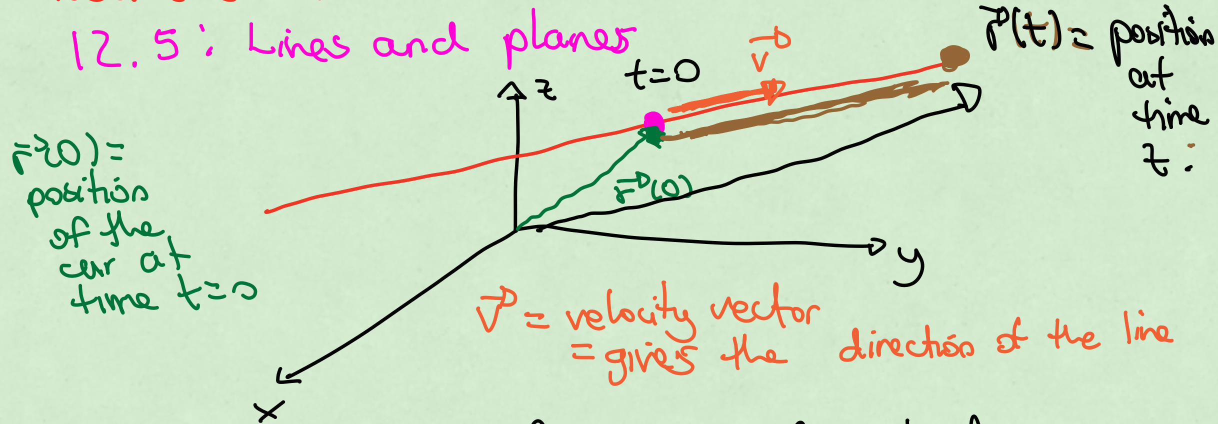


normal vector
$\vec{n}$

plane { normal vector + point := perpendicular to the plane

Lecture 5 (12.5 – 12.6)

12.5: Lines and planes



line: trajectory of a particle which moves with constant velocity

$$\text{speed} = \frac{\text{distance}}{\text{time}} \rightarrow \boxed{\text{distance} = \text{time} \cdot \text{speed}}$$

$$\boxed{\text{displacement} = \text{time} \cdot \text{velocity}}$$

$$\vec r(t) - \vec r(0) = \text{net displacement of the car}$$
$$= t\vec v$$

vector equation of a line is

$$\vec r(t) = \vec r(0) + t\vec v$$

Find the equation of a line if

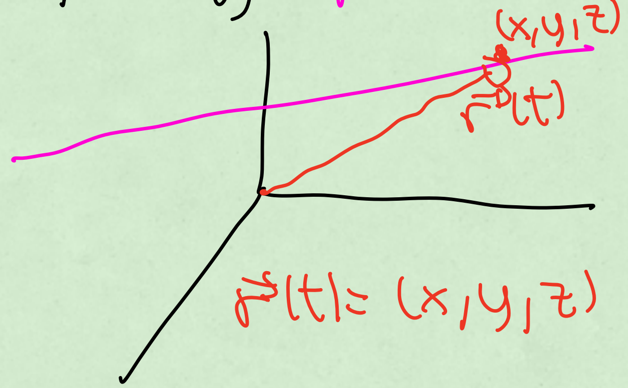
$$\vec r(0) = (1, -1, 5)$$
$$\vec v = (2, 7, 5)$$

$$\vec r(t) = \vec r(0) + t\vec v$$
$$\frac{d\vec r}{dt} = \vec v$$
$$\frac{d^2\vec r}{dt^2} = \vec 0 = \vec a$$

$$\vec{r}(t) = (1,-1,5) + t\,(2,7,5)$$

$$\vec{r}(t) = (1+2t,\ -1+7t,\ 5+5t)$$

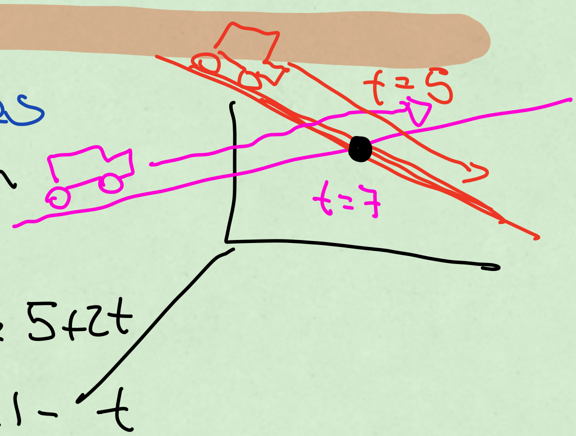
$$(x,y,z) = (1+2t,-1+7t,5+5t)$$ Parametric equation of line



$$\begin{cases} x = 1+2t \\ y = -1+7t \\ z = 5+5t \end{cases}$$

→ you would call this the parametric equation of a line (this is the version most used in this class)

Interaction of lines



Find if the lines with equations

$$\begin{cases} x = 2-t \\ y = 3t \\ z = 1+t \end{cases}, \quad \begin{cases} x = 5+2t \\ y = 1-t \\ z = 8+3t \end{cases}$$

we don't care if the cars collide, only if the paths intersect

set the equations equal to each other but change the name of "t" for one of the lines

$$\begin{cases} 2-t = 5+2s \\ 3t = 1-s \quad \longrightarrow \quad \boxed{s = 1-3t} \\ 1+t = 8+3s \end{cases}$$

$$\begin{cases} 2-t = 5+2(1-3t) \\ 1+t = 8+3(1-3t) \end{cases}$$

$$\begin{cases} 2-t = 5+2-6t \\ 1+t = 8+3-9t \end{cases}$$

$$\begin{cases} 5t = 5 \quad \rightarrow \quad t=1 \\ 10t = 10 \quad \rightarrow \quad t=1 \end{cases} \longrightarrow \quad \begin{aligned} s &= 1-3 \\ s &= -2 \end{aligned}$$

if you had found different values of "t" when solving both equations, then there would be no intersection.

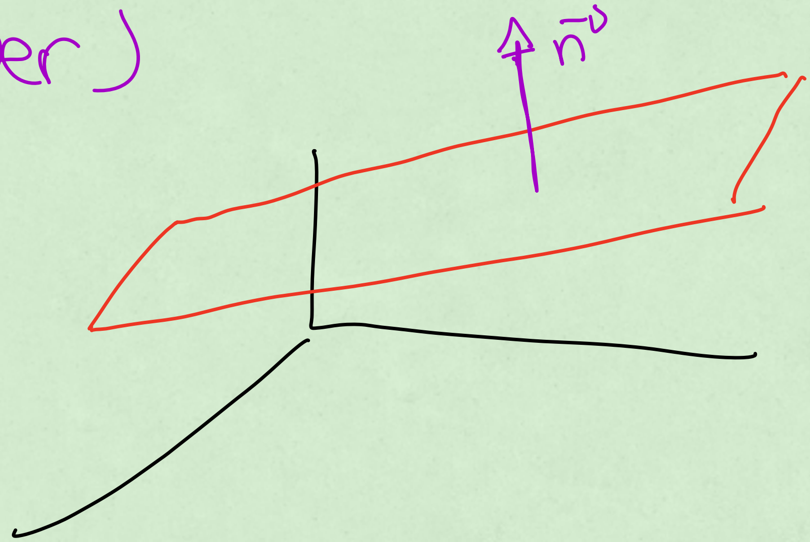
$t = 1$, $s = -2$

$$\begin{cases} x = 2 - t \\ y = 3t \\ z = 1 + t \end{cases} \qquad \begin{cases} x = 5 + 2s \\ y = 1 - s \\ z = 8 + 3s \end{cases}$$

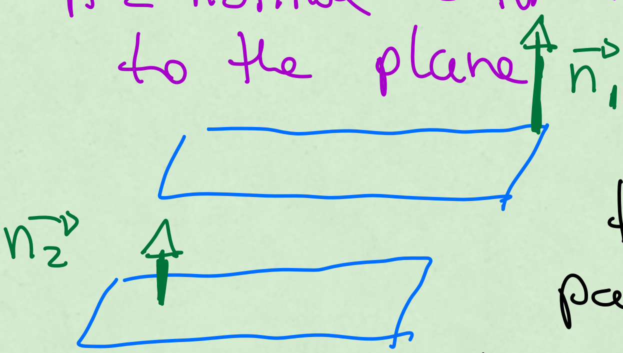
$$\underbrace{(1, 3, 2)}$$

where the lines intersect

Planes (infinite sheet of paper)

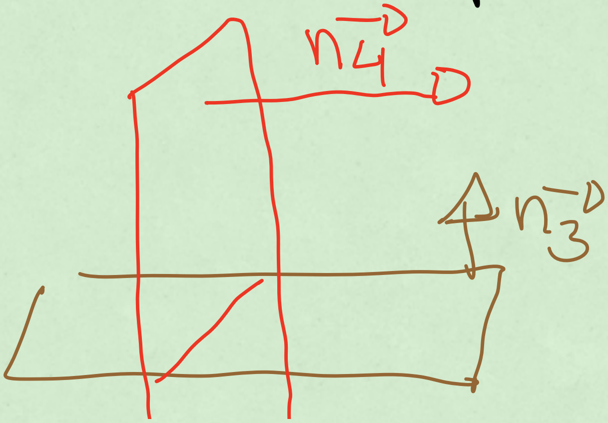


$\vec{n}$ = normal vector = vector perpendicular to the plane

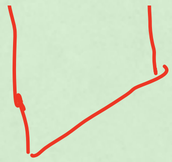


two planes are parallel if their normal vectors are parallel

$\vec{n_1} \times \vec{n_2} = \vec{0}$



two planes are perpendicular if their normal

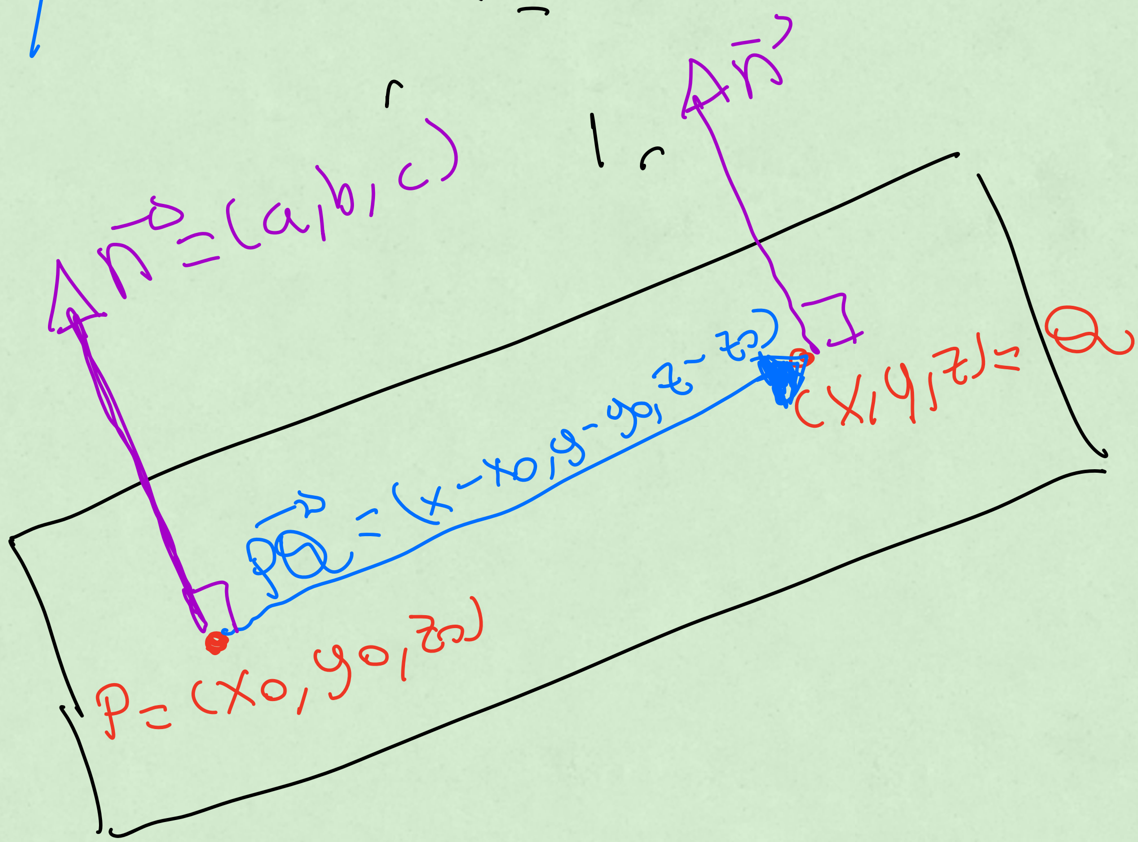


vectors are
perpendicular

$$\vec{n_3} \cdot \vec{n_4} = 0$$

---

equation of a plane



$\vec{n} = (a, b, c)$ (entries of normal vector)

(x_0, y_0, z_0) (known point on

the plane)

to find the equation of a plane you need the normal vector and a point on the plane

$$\vec{n} \cdot \vec{PQ} = 0$$

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) =$$

$$a(x - x_0) + b(y - y_0) +$$

$$c(z - z_0) = 0$$

$$ax - ax_0 + by - by_0$$
$$+ cz - cz_0 = 0$$

equation plane

$$\boxed{ax + by + cz = ax_0 + by_0 + cz_0}$$

example

$$\vec{n} = (\overset{a}{-1}, \overset{b}{2}, \overset{c}{5})$$

point $(x_0, y_0, z_0) = (2, 3, 7)$

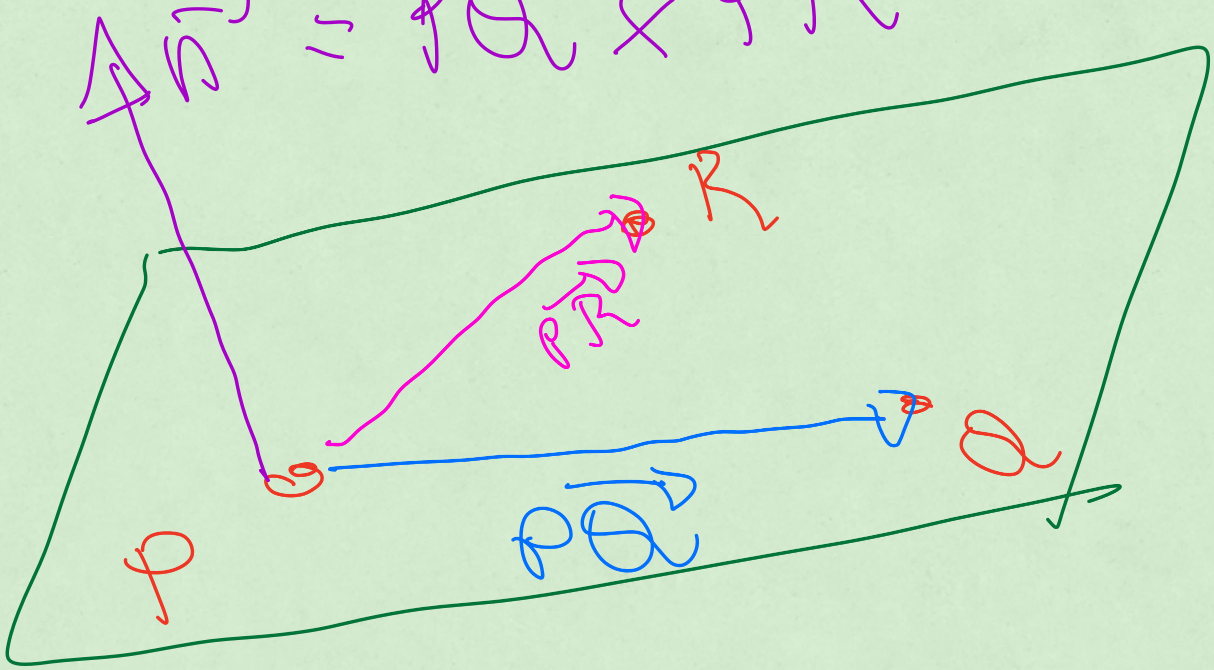
$$-x + 2y + 5z$$

$$= -2 + 6 + 35$$

$$\boxed{-x + 2y + 5z = 39}$$

coefficients multiplying x, y, z are the entries of the normal vector

$$\vec{n} = \vec{PQ} \times \vec{PR}$$



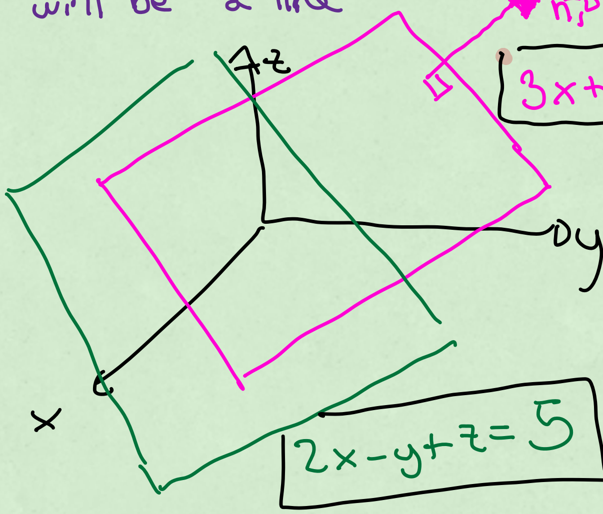
So you can find $\vec{n}$ if you are given 3 points on the plane

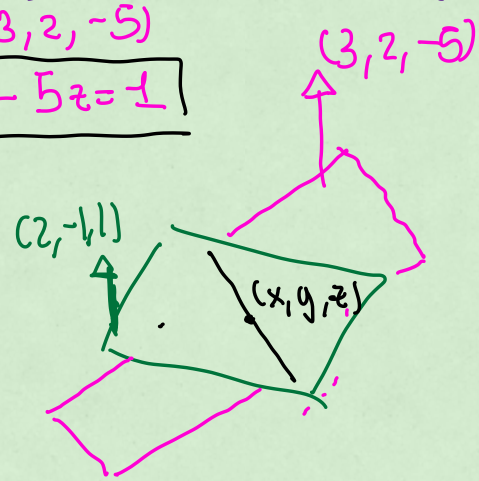
Lecture 6 (12.6, 13.1, 13.2)

If two planes are not parallel, their intersection will be a line

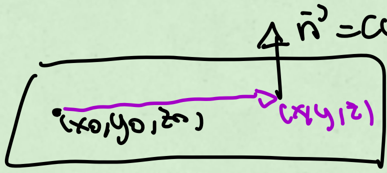
$\vec{n}_P = (3, 2, -5)$

$\boxed{3x + 2y - 5z = 1}$





$\boxed{2x - y + z = 5}$



$\boxed{ax + by + cz = \boxed{ax_0 + by_0 + cz_0}}$

called "d"

$ax + by + cz = d$

changing this number creates parallel planes

line of intersection:
the points (x, y, z) where both equations are satisfied.

$$\begin{cases} 3x + 2y - 5z = 1 \\ 2x - y + z = 5 \end{cases}$$

3 unknowns x, y, z
2 equations

write two of the variables in terms of the remaining

$\boxed{z = 5 - 2x + y}$

$$3x + 2y - 5(5 - 2x + y) = 1$$

$$13x - 3y = 26$$

$$13x = 26 + 3y$$

$$\boxed{x = 2 + \frac{3}{13}y}$$

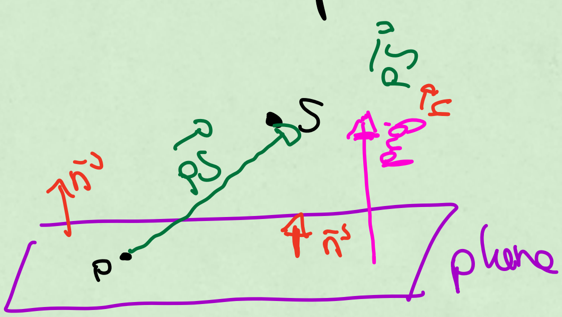
$$z = 5 - 2\left(2 + \frac{3}{13}y\right) + y$$

$$\boxed{z = 1 + \frac{7y}{13}}$$

$$\begin{cases} x = 2 + \frac{3}{13}y \\ y = t \\ z = 1 + \frac{7y}{13} \end{cases} \longrightarrow \begin{cases} x = 2 + \frac{3}{13}t \\ y = t \\ z = 1 + \frac{7t}{13} \end{cases}$$

parametric equation of a line

distance point to a plane



P = point on the plane

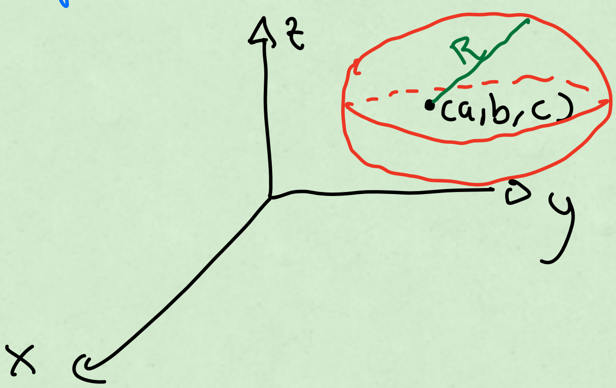
distance from point S to plane

$$= \left|\operatorname{proj}_{\vec{n}} \vec{PS}\right|$$

$$= \left|\frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|}\right|$$

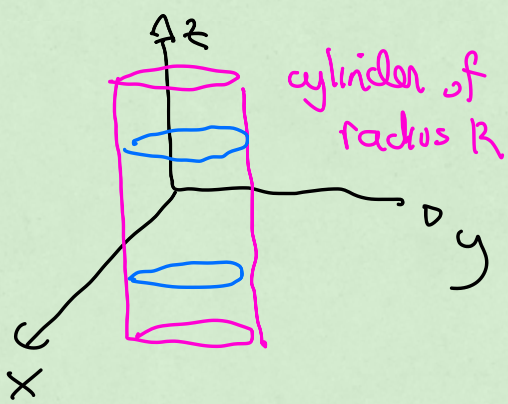
$\boxed{12.6}$ (visual recognition of certain equations)

sphere: $(x-a)^2+(y-b)^2+(z-c)^2=R^2$



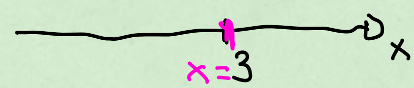
radius R

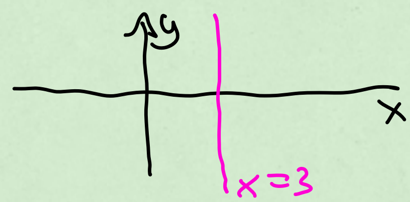
$x^2+y^2=R^2$

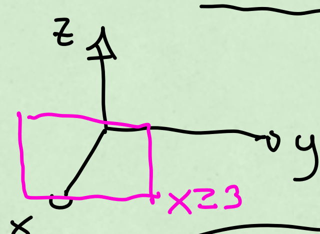


cylinder of radius R

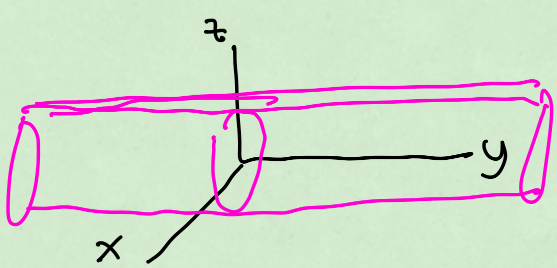
$x=3$



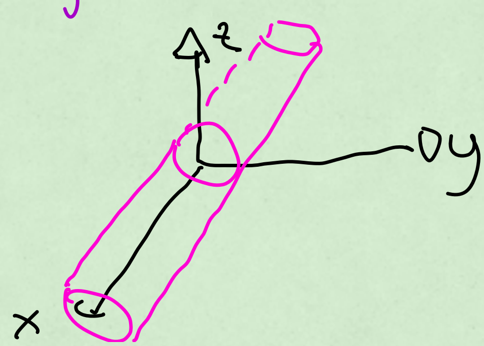




$x^2+z^2=R^2$

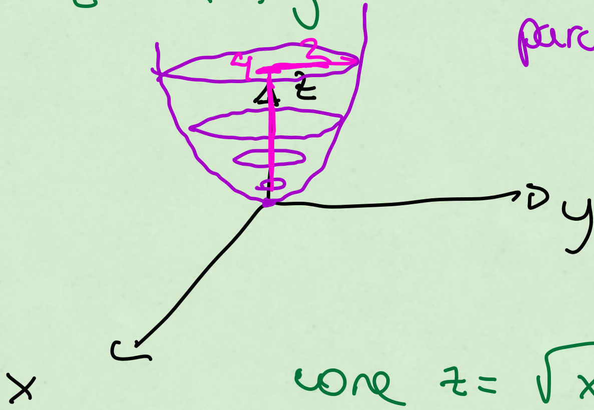


$y^2+z^2=R^2$

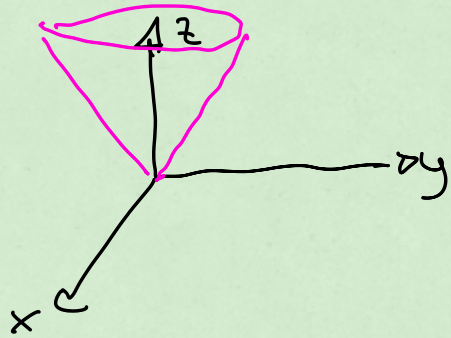


$z = x^2 + y^2$ (circle radius $\sqrt{z}$)

paraboloid



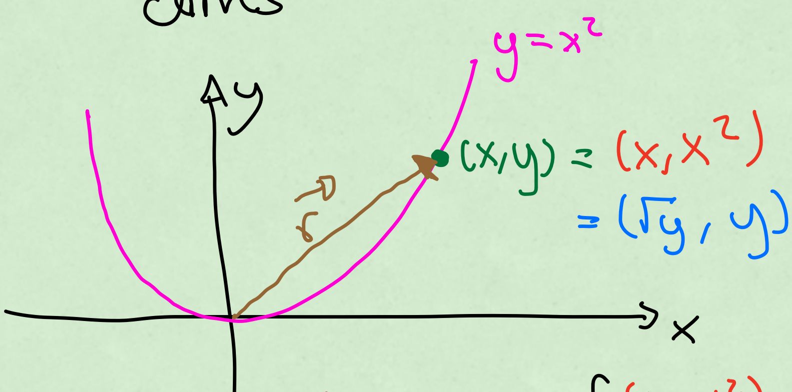
cone $z = \sqrt{x^2 + y^2}$



Chapter 13 (13.1, 13.2, 13.3)

$\underbrace{\text{Chapter 13}}_{\text{curves}}$

$\vec{r}$ = position vector from origin to some point on the curve



$\vec{r} = (x, y) = \{ (x, x^2) = \vec{r}(x) \to \vec{r}(t) = (t, t^2)$

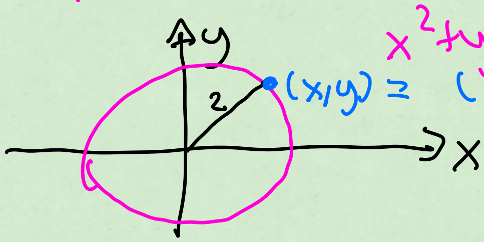
$\{ (\sqrt{y}, y) = \vec{r}(y) \to \vec{r}(t) = (\sqrt{t}, t)$

In general, you write this as $\vec{r}(t)$, so that you can think of "t" as behaving as the time variable

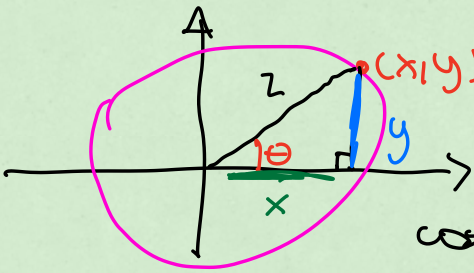
---

Important example

$x^2 + y^2 = 4$



$(x, y) = (x, \sqrt{4 - x^2})$ fine but not ideal since you need $\pm$ for square root

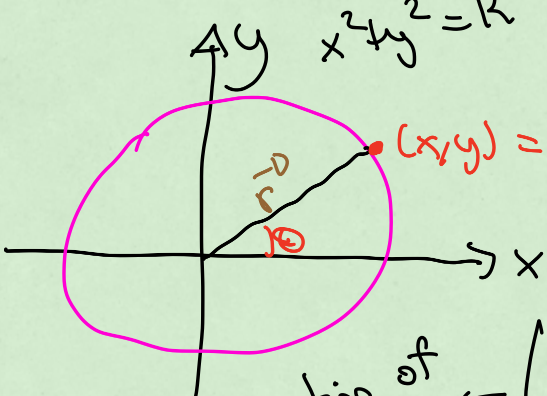


$(x, y) = (2\cos\theta, 2\sin\theta)$

$0 \le \theta \le 2\pi$

$\cos\theta = \frac{x}{2} \qquad \sin\theta = \frac{y}{2}$

$x^2 + y^2 = R^2$

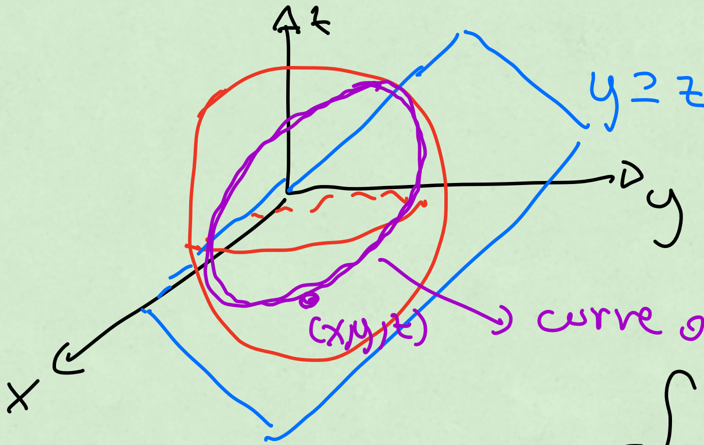


$(x, y) = (R\cos\theta, R\sin\theta)$

$\vec{r}(\theta) = (R\cos\theta, R\sin\theta)$

equation of a circle of radius R $\leftarrow$ $\boxed{\vec{r}(t) = (R\cos t, R\sin t)}$

Difficult example $x^2 + \frac{y^2}{2} + \frac{z^2}{2} = 4$



→ curve of intersection

→ curves are obtained from intersecting two surfaces

$$\begin{cases} x^2 + \frac{y^2}{2} + \frac{z^2}{2} = 4 \\ y = z \end{cases}$$

$$x^2 + \frac{y^2}{2} + \frac{y^2}{2} = 4$$

$$x^2 + y^2 = 4$$

previous example → $x = 2\cos t$, $y = 2\sin t$

$$\vec{r} = (x, y, z) = (2\cos t, 2\sin t, 2\sin t)$$

$$\boxed{\vec{r}(t) = (2\cos t, 2\sin t, 2\sin t)}$$

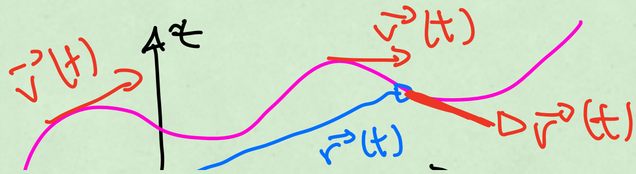
↓

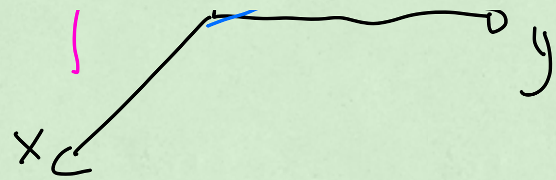
position vector as a function of t.

$$\vec{v} = \frac{d\vec{r}}{dt} = (-2\sin t, 2\cos t, 2\cos t)$$

↓

velocity vector





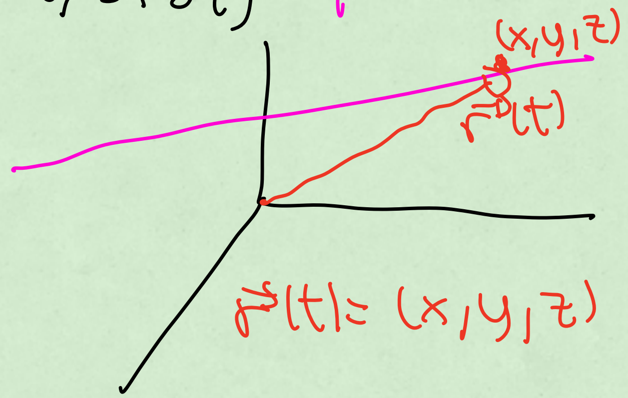
$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = (-2\cos t, -2\sin t, -2\sin t)$$

acceleration vector

$\vec{r}(t) = (1,-1,5) + t\,(2,7,5)$

$\vec{r}(t) = (\,1+2t,\ -1+7t,\ 5+5t)$

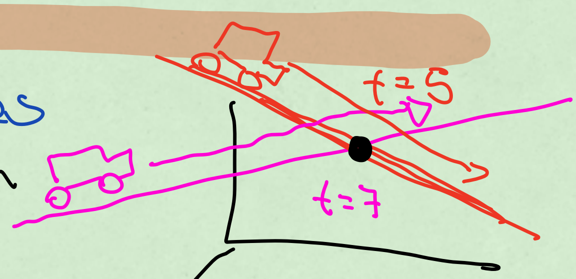
$(x,y,z) = (1+2t, -1+7t, 5+5t)$ Parametric equation of line



$$\begin{cases} x = 1+2t \\ y = -1+7t \\ z = 5+5t \end{cases}$$

→ you would call this the parametric equatión of a line (this is the versión most used in this class)

Interaection of lines



Find if the lines with equatións

$$\begin{cases} x = 2-t \\ y = 3t \\ z = 1+t \end{cases} , \qquad \begin{cases} x = 5+2t \\ y = 1-t \\ z = 8+3t \end{cases}$$

we don't care if the cars collide, only if the paths intersect

set the equations equal to each other but change the name of "t" for one of the lines

$$\begin{cases} 2-t = 5+2s \\ 3t = 1-s \rightarrow \boxed{s=1-3t} \\ 1+t = 8+3s \end{cases}$$

$$\begin{cases} 2-t = 5+2(1-3t) \\ 1+t = 8+3(1-3t) \end{cases}$$

$$\begin{cases} 2-t = 5+2-6t \\ 1+t = 8+3-9t \end{cases}$$

$$\begin{cases} 5t = 5 \rightarrow t=1 \\ 10t = 10 \rightarrow t=1 \end{cases} \rightarrow s=1-3 \quad s=-2$$

if you had found different values of "t" when solving both equations, then there would be no intersection.

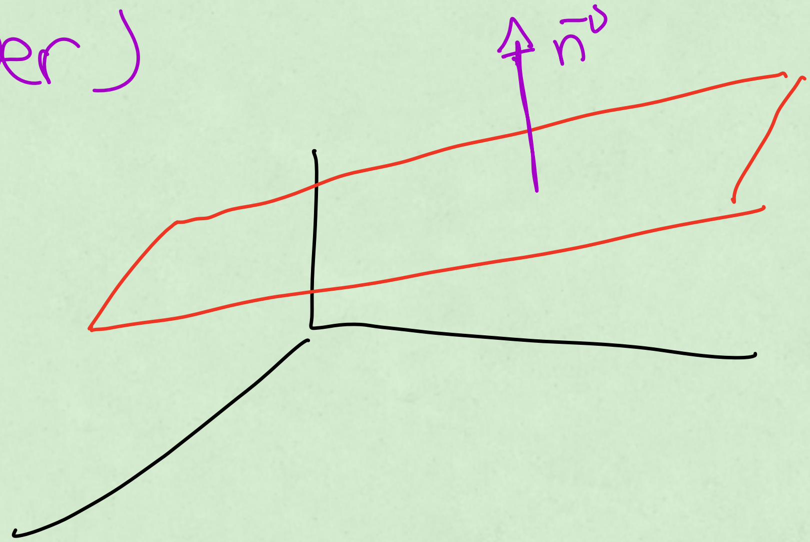
$t = 1$, $s = -2$

$$\begin{cases} x = 2 - t \\ y = 3t \\ z = 1 + t \end{cases} \qquad \begin{cases} x = 5 + 2s \\ y = 1 - s \\ z = 8 + 3s \end{cases}$$

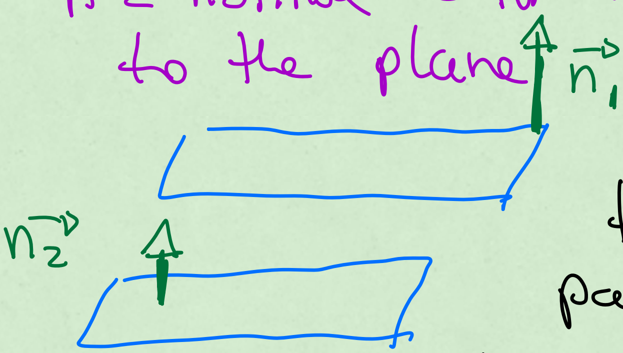
$(1, 3, 2)$

where the lines intersect

Planes (infinite sheet of paper)

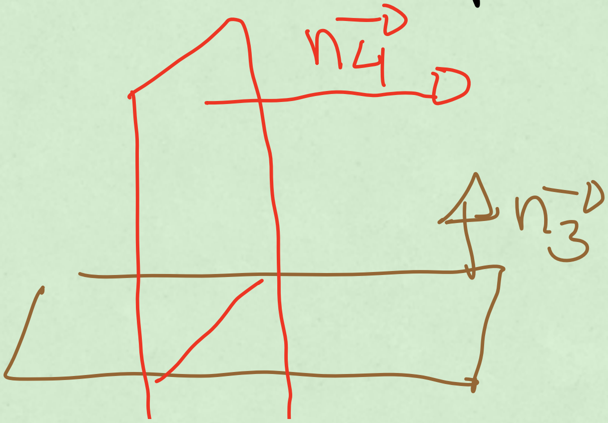


$\vec{n}$ = normal vector = vector perpendicular to the plane



two planes are parallel if their normal vectors are parallel

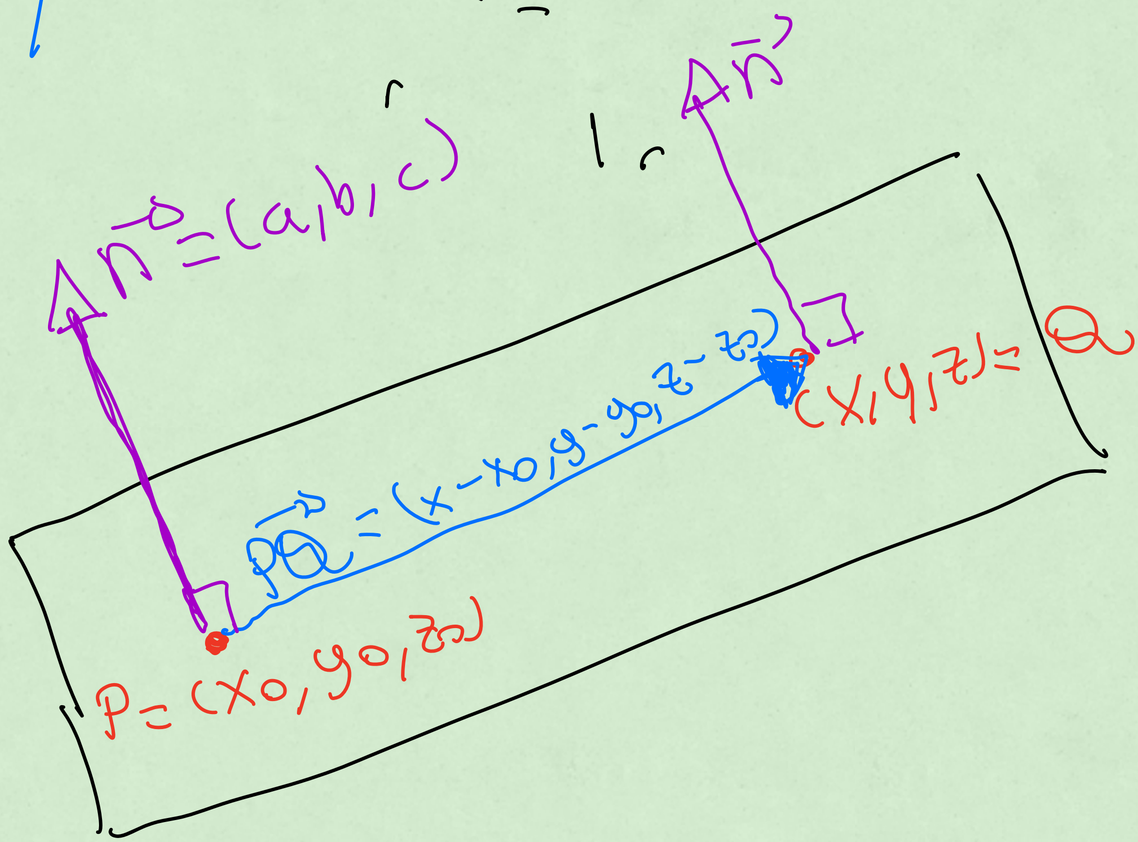
$\vec{n_1} \times \vec{n_2} = \vec{0}$



two planes are perpendicular if their normal vectors are perpendicular

$$\vec{n_3} \cdot \vec{n_4} = 0$$

equation of a plane



$\vec{n} = (a,b,c)$ (entries of normal vector)

(x_0, y_0, z_0) (known point on

the plane)

to find the equation of a plane you need the normal vector and a point on the plane

$$\vec{n} \cdot \overrightarrow{PQ} = 0$$

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) =$$

$$a(x - x_0) + b(y - y_0) +$$

$$c(z - z_0) = 0$$

$$ax - ax_0 + by - by_0$$
$$+ cz - cz_0 = 0$$

equation plane

$$\boxed{ax + by + cz = ax_0 + by_0 + cz_0}$$

example

$$\vec{n} = (\overset{a}{-1}, \overset{b}{2}, \overset{c}{5})$$

point $(x_0, y_0, z_0) = (2, 3, 7)$

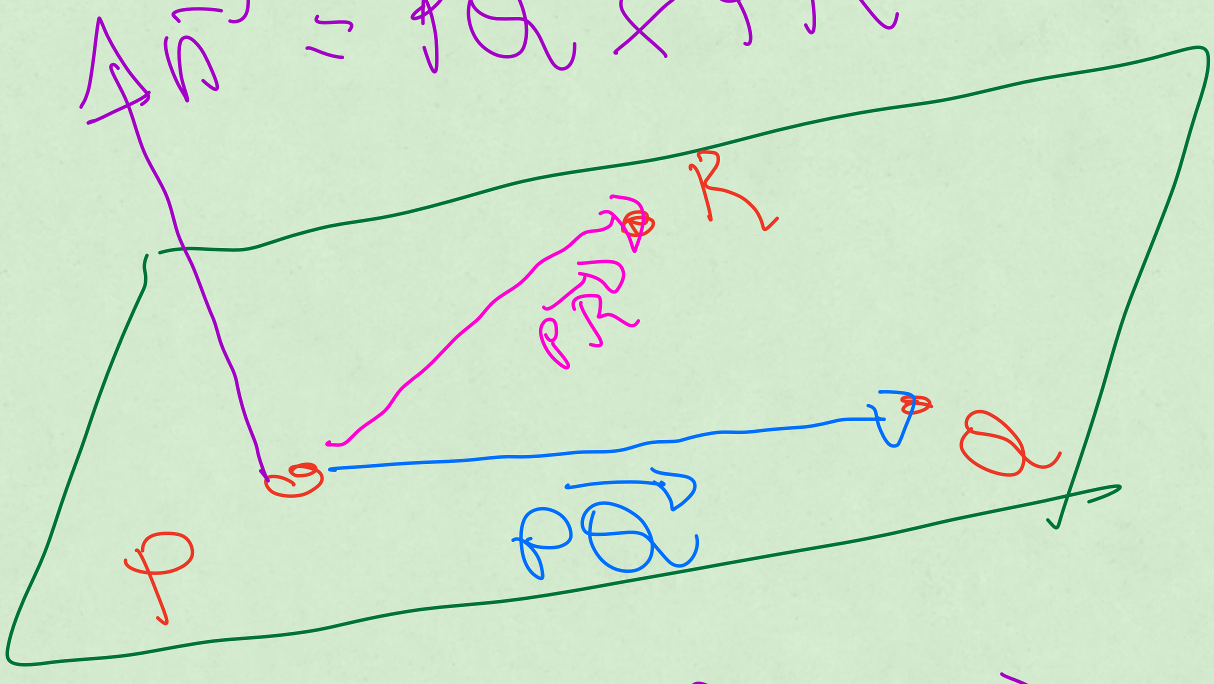
$$-x + 2y + 5z$$

$$= -2 + 6 + 35$$

$$\boxed{-x + 2y + 5z = 39}$$

coefficients multiplying x, y, z are the entries of the normal vector

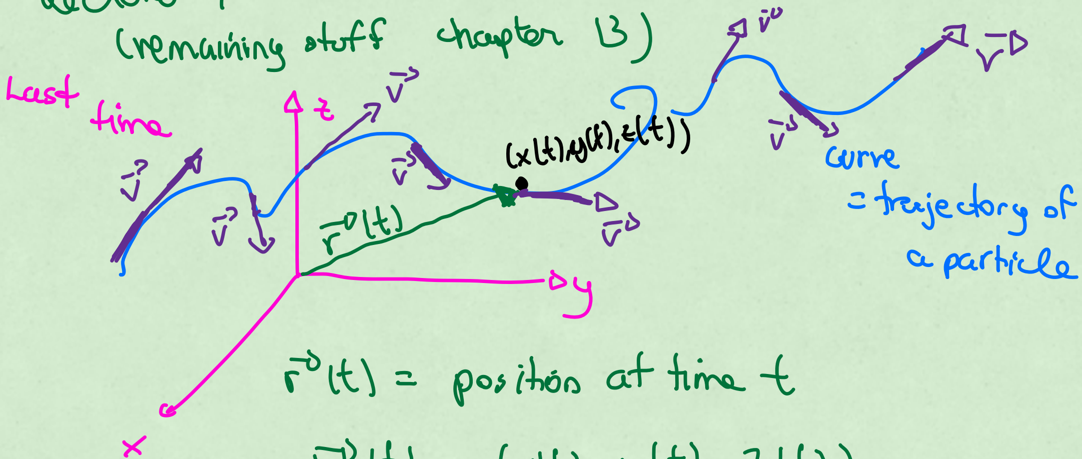
$\vec{n} = \vec{PQ} \times \vec{PR}$



so you can find $\vec{n}$ if you are given 3 points on the plane

Lecture 7
(remaining stuff chapter 13)

Last time



$\vec{r}(t)$ = position at time t

$$\underbrace{\vec{r}(t)}_{\text{position vector}} = (x(t),\ y(t),\ z(t))$$

velocity vector $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right)$

speed $= |\vec{v}(t)| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}$

velocity ⇝ refers to a vector

speed ⇝ magnitude of velocity ⇝ non negative number (non negative scalar)

accelaration

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$$

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

example:

curve

a) $\vec{r}(t) = (t^2, \sin t, e^t)$

Find the velocity and acceleration of the curve

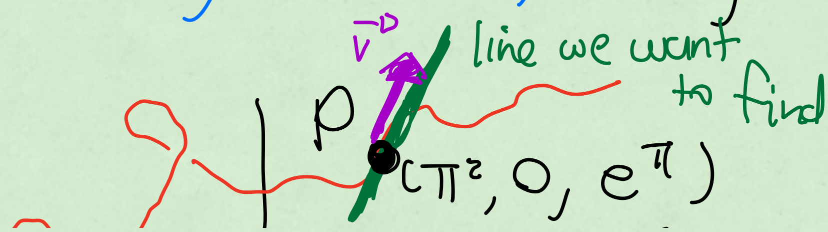
b) Find the tangent line to this curve at the point $P = (\pi^2, 0, e^\pi)$

---

a) $\vec{v}(t) = (2t, \cos t, e^t)$

$\vec{a}(t) = (2, -\sin t, e^t)$

b)



$\vec{r}(t) = (t^2, \sin t, e^t)$

$(t^2, \sin t, e^t) = (\pi^2, 0, e^\pi)$

$\Rightarrow \boxed{t = \pi}$

$\vec{v}(t) = (2t, \cos t, e^t)$

we need this at the time we go through P

$\vec{v}(\pi) = \underbrace{(2\pi, -1, e^\pi)}_{\text{direction of the tangent line}}$

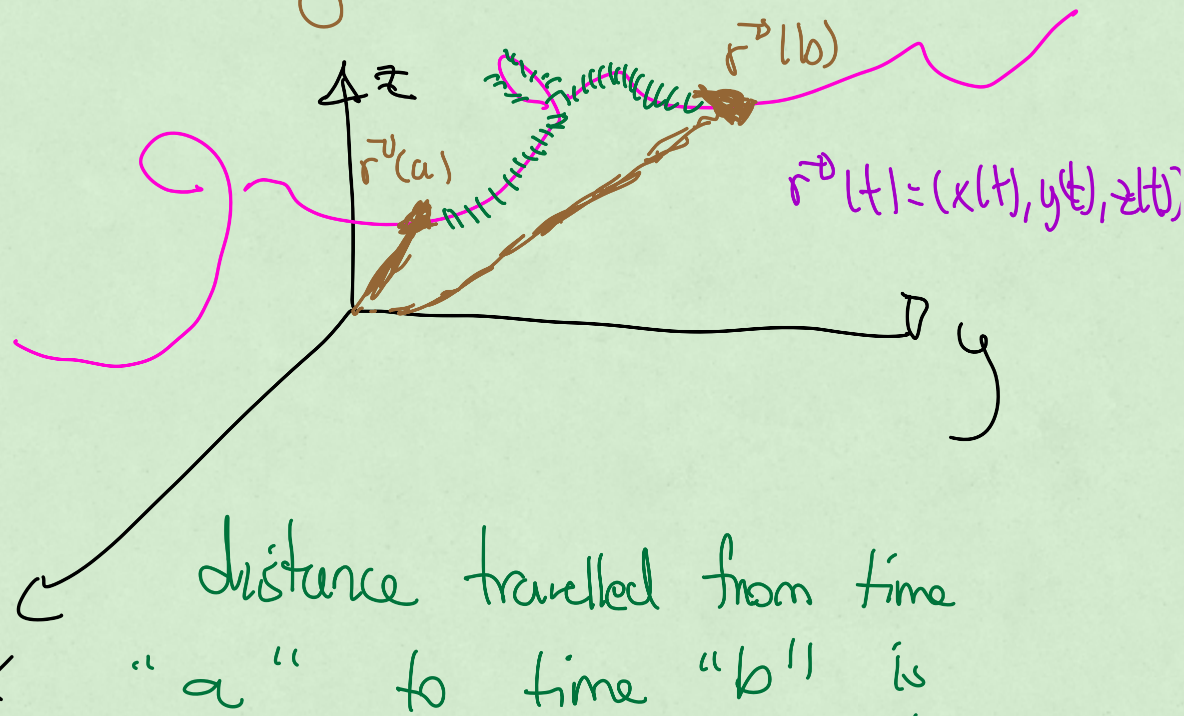
equation line

$(x, y, z) = \text{point} + u \ \text{direction}$

$(x, y, z) = (\pi^2, 0, e^\pi) + u(2\pi, -1, e^\pi)$

parametric form
$$\begin{cases} x = \pi^2 + 2\pi u \\ y = 0 - u \\ z = e^\pi + e^\pi u \end{cases}$$

Length of a curve and arclength



distance travelled from time "a" to time "b" is the length of the curve between the points $\vec{r}(a)$ and $\vec{r}(b)$

" $\text{speed} = \dfrac{\text{distance}}{\text{time}}$ "

" distance = speed · time "

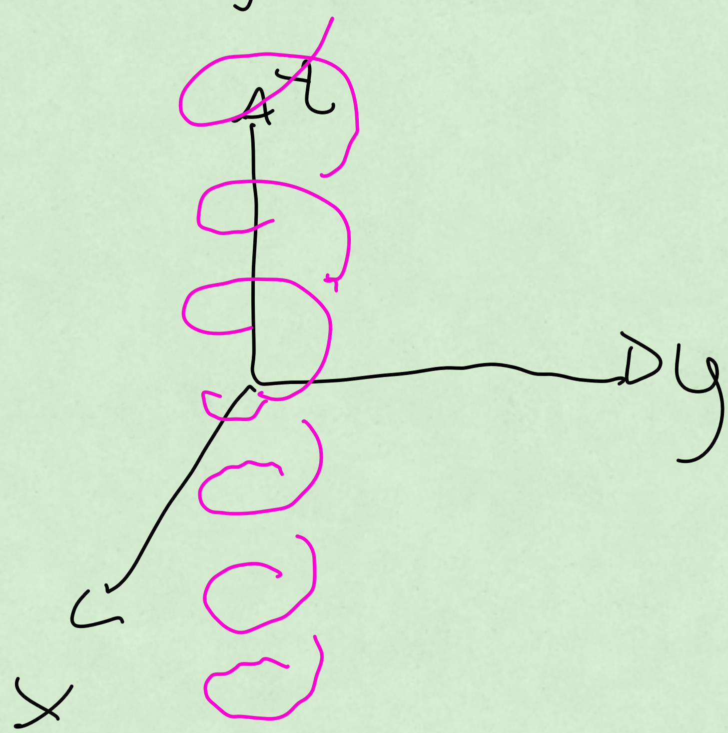
Length or distance from time

t = a to time t = b is

$$\boxed{\text{length} = \int_a^b \|\vec{v}(t)\| \, dt}$$

example: helix

$$\vec{r}(t) = (\cos t, \sin t, t)$$



Find length of the helix from time 0 to time 2π.

$$\vec{r}(t) = (\cos t, \sin t, t)$$

$$\vec{v}(t) = (-\sin t, \cos t, 1)$$

$$|\vec{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$$

$$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$|\vec{v}(t)| = \sqrt{2}$$

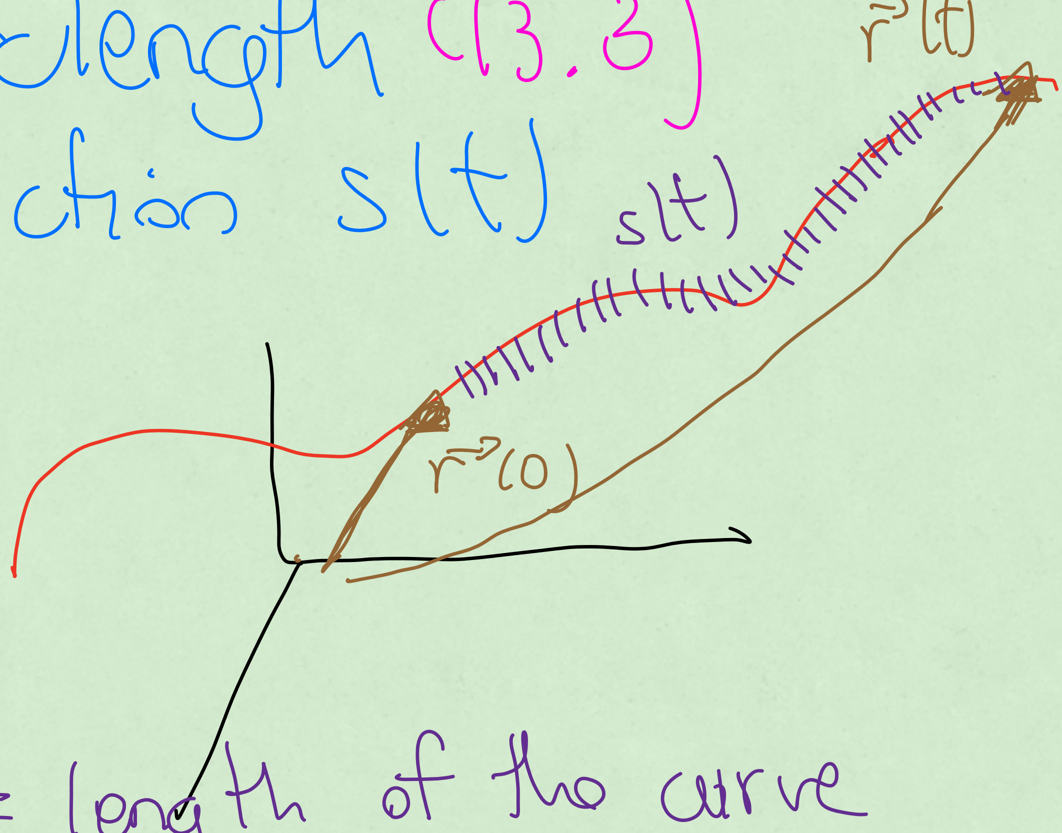
$$\text{length} = \int_0^{2\pi} |\vec{v}(t)|\,dt$$

$$\text{length} = \int_0^{2\pi} \sqrt{2}\, dt$$

$$\text{length} = 2\sqrt{2}\,\pi$$

arclength (13.3)

function $s(t)$



$s(t)$ = length of the curve from time 0 to time "t"

$$s(t) = \int_0^t \|\vec{v}(t)\|\,dt$$

now the upper bound is the variable "t"

example:

$\vec{r}(t) = (1, e^t, 3e^t)$

find arclength function $s(t)$.

$\vec{v}(t) = (0, e^t, 3e^t)$

$$|\vec{v}(t)| = \sqrt{0^2 + e^{2t} + 9e^{2t}}$$

$$|\vec{v}(t)| = \sqrt{10e^{2t}}$$

$$|\vec{v}(t)| = \sqrt{10}\, e^t$$

$$s(t) = \int_0^t |\vec{v}(t)|\, dt$$

$$s(t) = \int_0^t \sqrt{10}\, e^t\, dt$$

$$s(t) = \sqrt{10}\, e^t \Big|_0^t$$

$$\boxed{s(t) = \sqrt{10}\,(e^t - 1)}$$

length of the curve between time "0" and time "t"

arclength parametrization of the curve; here you reverse the roles and think of "t" as a function of "s"

$$s = \sqrt{10}\,(e^t - 1)$$

$$\frac{s}{\sqrt{10}} = e^t - 1$$

$$\frac{s}{\sqrt{10}} + 1 = e^t$$

$$\boxed{\ln\left(\frac{s}{\sqrt{10}}+1\right) = t}$$

so you use this to rewrite $\vec{r}(t)$ as $\vec{r}(s)$, so you specify the position vector by "s".

$$\vec{r}(t) = \left(1, e^t, 3e^t\right)$$

$$\vec{r}(s) = \left(1, e^{\ln\left(\frac{s}{\sqrt{10}}+1\right)}, 3e^{\ln\left(\frac{s}{\sqrt{10}}+1\right)}\right)$$

$$\vec{r}(s) = \left(1, \frac{s}{\sqrt{10}}+1, 3\left(\frac{s}{\sqrt{10}}+1\right)\right)$$

Finding arc length parametrization:

1. Find $s(t)$
2. use that formula to write "t" in terms of "s"
3. substitute "t" in your expression for $\vec{r}(t)$ to get $\vec{r}(s)$

---

example: helix

$$\vec{r}(t) = (\cos t, \sin t, t)$$

$$\vec{v}(t) = (-\sin t, \cos t, 1)$$

$$|\vec{v}(t)| = \sqrt{2}$$

$$s(t) = \int_0^t |\vec{v}(t)|\, dt$$

$$s(t) = \int_0^t \sqrt{2}\, dt$$

$$s(t) = \sqrt{2}\,t$$

$$s = \sqrt{2}\,t$$

$$\boxed{t = \frac{s}{\sqrt{2}}}$$

$$\vec{r}(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)$$

---

section 13.4

tangent vector $\vec{T}(t)$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

example $\vec{r}(t) = (1, e^t, 3e^t)$

$\vec{v}(t) = (0, e^t, 3e^t)$

$|\vec{v}(t)| = \sqrt{10}\, e^t$

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

$$\vec{T}(t) = \frac{(0, e^t, 3e^t)}{\sqrt{10}\, e^t}$$

$$\vec{T}(t) = \left(0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

section 13.2

if you are given $\vec{v}(t)$, you can find $\vec{r}(t)$ by integrating $\vec{v}(t)$ with respect to "t"

---

say if $\vec{v}(t) = (t, e^t, t^2)$

and $\vec{r}(0) = (0, 1, 2)$

Find $\vec{r}(t)$

$$\vec{r}(t) = \left(\frac{t^2}{2} + c_1, \; e^t + c_2, \; \frac{t^3}{3} + c_3\right)$$

$t = 0$

$$\vec{r}(0) = (c_1, 1 + c_2, c_3)$$

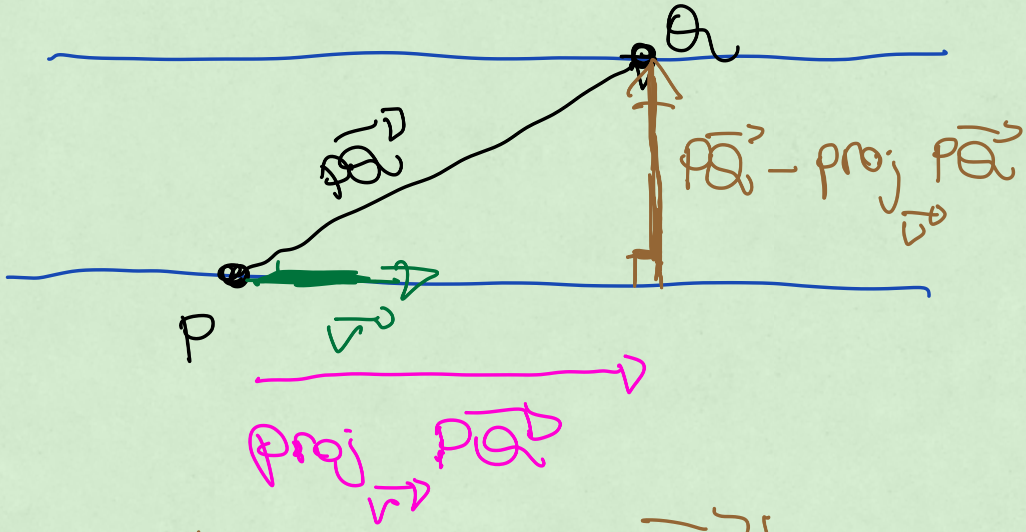
$$= (0, 1, 2)$$

$c_1 = 0$, $c_2 = 0$, $c_3 = 2$

$$\vec{r}(t) = \left(\frac{t^2}{2}, e^t, \frac{t^3}{3} + 2\right)$$

---

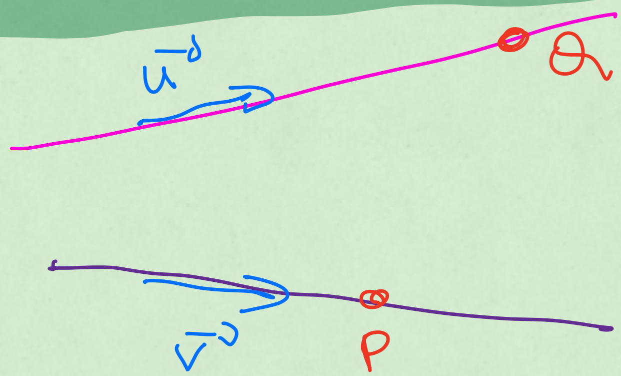
distance between parallel lines (not on the exam)



$$\text{distance} = \left| \vec{PQ} - \mathrm{proj}_{\vec{v}} \vec{PQ} \right|$$

---

skew lines



distance (not on the exam) $= \dfrac{\left| \vec{PQ} \cdot (\vec{u} \times \vec{v}) \right|}{\left| \vec{u} \times \vec{v} \right|}$