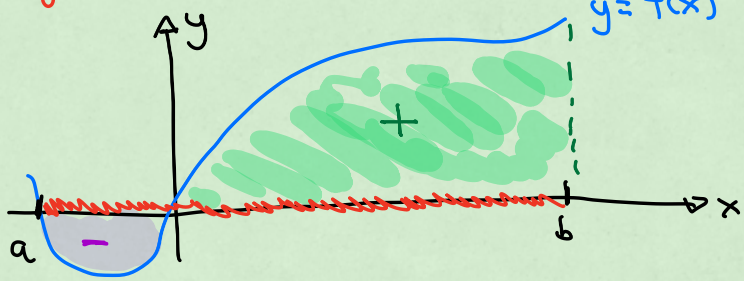
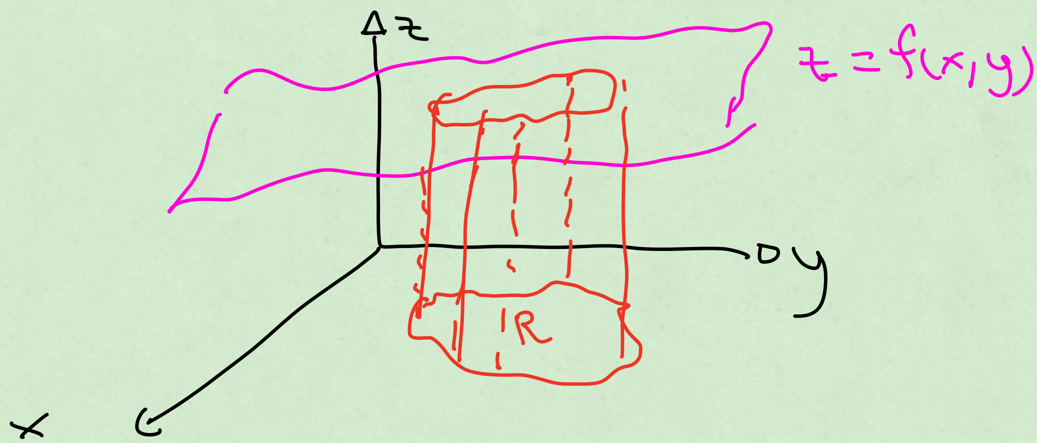
Lecture 15 (15.1 - 15.2)

Integrals

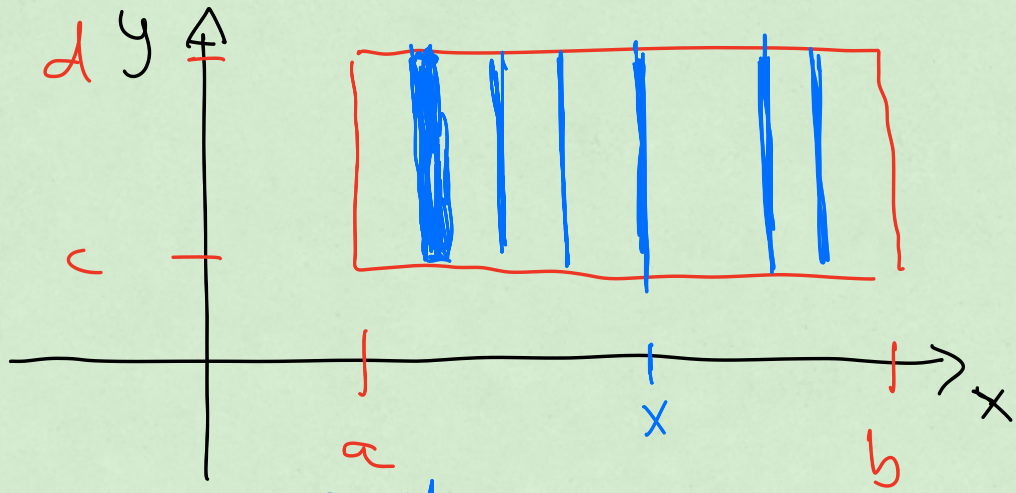


$\int_a^b f(x)dx$ = "net" area between the graph and the x-axis



$\iint_R f(x,y)\,dA$ = "net" volume between the graph of the function and the region R on the xy plane

Vertical slices (or vertical cuts)

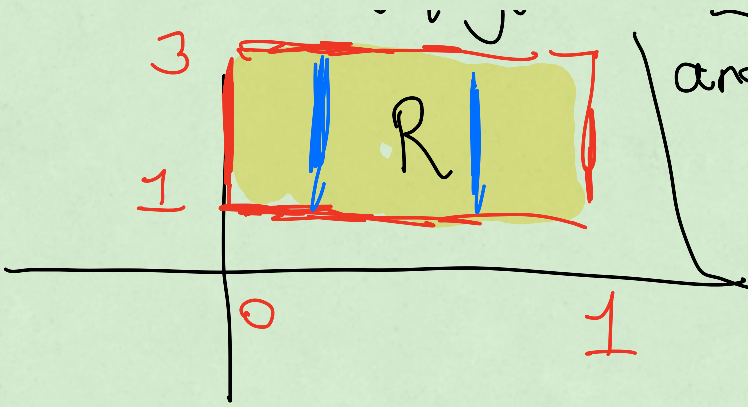


on each of these vertical slices
only "y" is changing, x is fixed

$$A(x) = \int_c^d f(x,y)\,dy$$ = area of the yellow slice from geogebra animation

volume = $$\int_a^b A(x)\,dx = \underbrace{\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx}_{(*)}$$

example
$$f(x,y) = 3xy^2 - x.$$



analogue of

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\iint_R (3xy^2 - x)\, dA$$

vertical cuts since "y" is integrated first and y axis is vertical ones

(*)

$$= \int_0^1 \left[\int_1^3 (3xy^2 - x)\, dy \right] dx$$

$$\int y^2\, dy = \frac{y^3}{3}$$

= partial integral

$$= \int_0^1 \left[xy^3 - xy \right]_{y=1}^{y=3} dx$$

$$\int 1\, dy = y$$

$$\int 3\pi y^2\, dy = \pi y^3$$

$$= \int_0^1 \left(27x - 3x - (x - x) \right) dx$$

$$= \int_0^1 24x \, dx$$

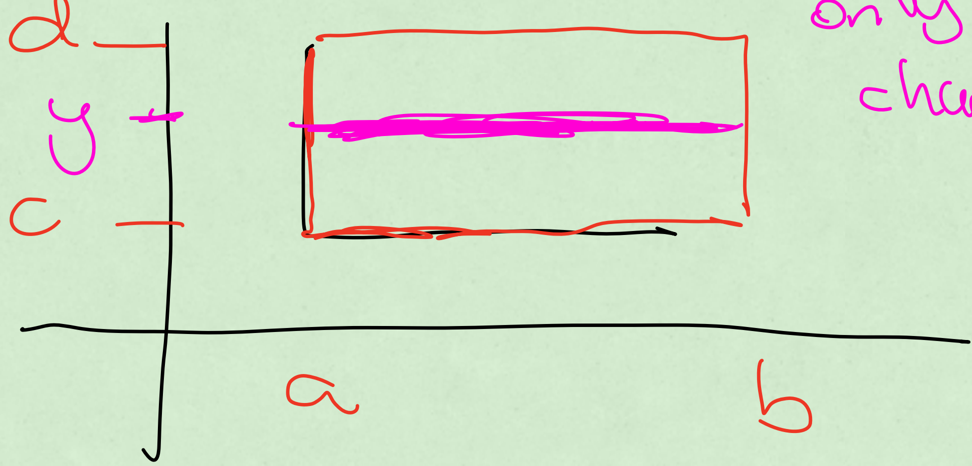
$$= 12x^2 \Big|_{x=0}^{x=1}$$

$$= \boxed{12}$$

$$3\cancel{x}y^2 - x \longrightarrow \boxed{\text{integrate } y} \longrightarrow 24x \longrightarrow \boxed{\text{integrate } x} \longrightarrow 12$$

Horizontal Slice

"y" is fixed only x changes



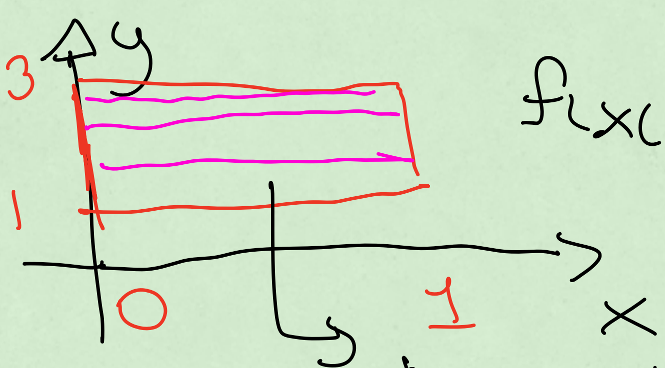
h

$$A(y) = \int_a f(x,y)\,dx$$ = area of the horizontal slice

volume = $$\int_c^d A(y)\,dy$$

$$= \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

Back to our example



$$f(x,y) = 3xy^2 - x$$

horizontal lines (thus parallel to x axis)

—

$$\int_1^3 \left[\int_0^1 (3xy^2 - x)\,dx\right] dy$$

$$= \int_1^3 \left(3y^2 \frac{x^2}{2} - \frac{x^2}{2}\right)\Big|_{x=0}^{x=1} dy$$

$$= \int_1^3 \left(3y^2 \cdot \frac{1}{2} - \frac{1}{2} - (0-0)\right) dy$$

$$= \int_1^3 \left(\frac{3y^2}{2} - \frac{1}{2}\right) dy$$

$$= \frac{y^3}{2} - \frac{y}{2}\Big|_{y=1}^{y=3}$$

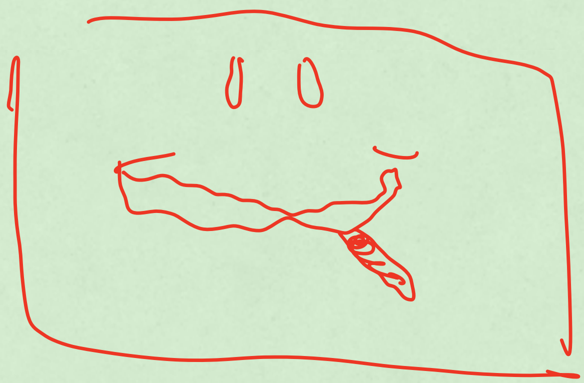
$$= \frac{27}{2} - \frac{3}{2} - \left(\frac{1}{2} - \frac{1}{2}\right)$$

$$= \frac{24}{2}$$

$$= \boxed{12}$$

$3xy^2 - x$ $\xrightarrow{\text{integrate } x}$ $\boxed{}$ $\to$ $\frac{3y^2}{2} - \frac{1}{2}$ $\xrightarrow{\text{integrate } y}$ $\boxed{}$ $\to 12$

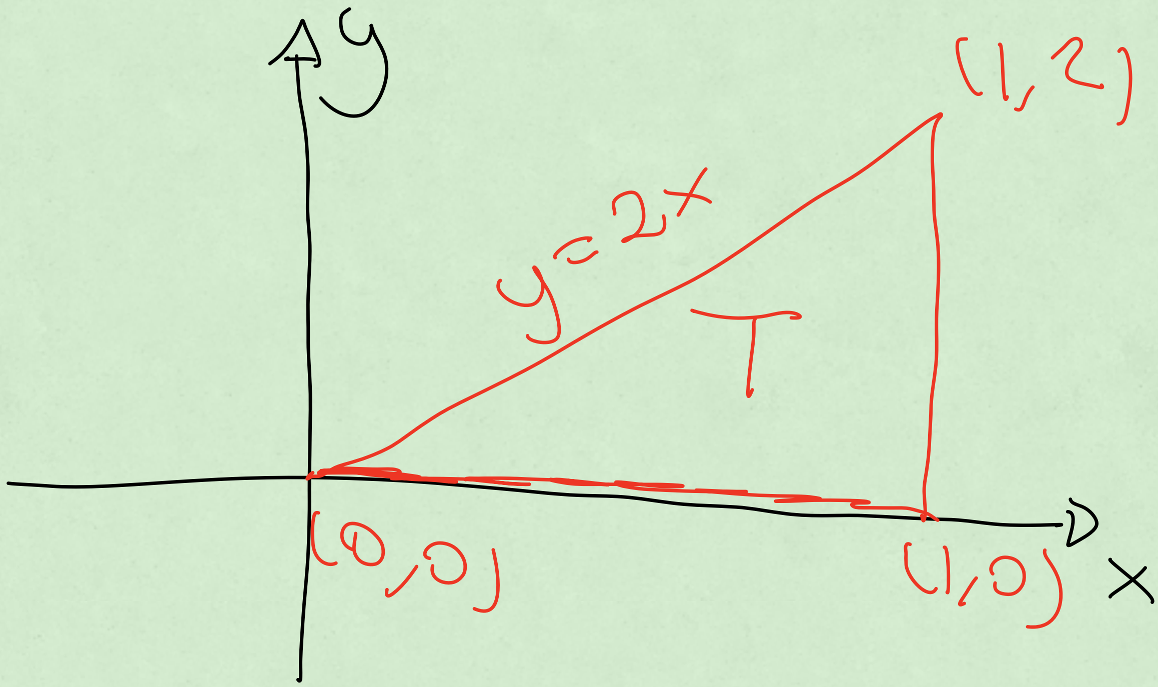
Same answer!



Fubini's Theorem

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

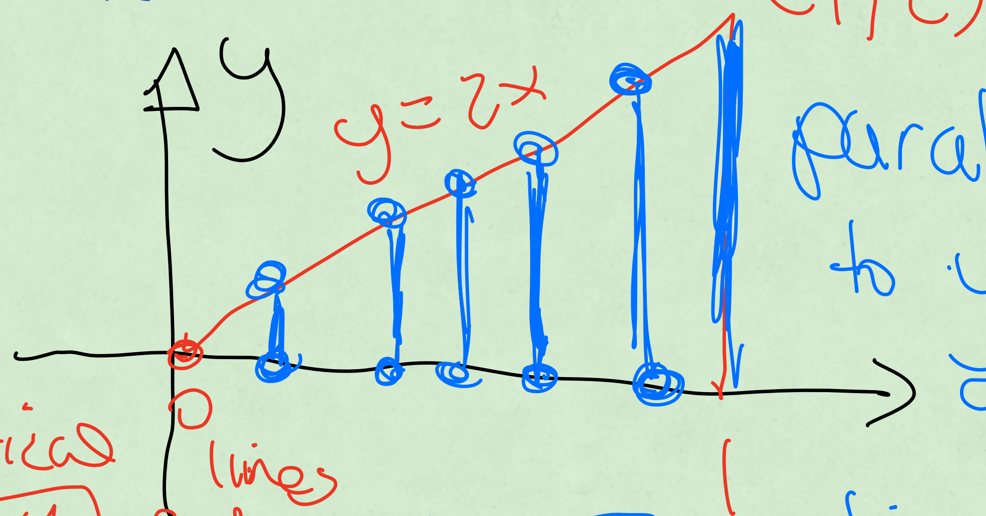
What if we don't have a rectangle?



$f(x,y) = x$, find.

$$\iint_T x \, dA$$

vertical slice



parallel to y axis

vertical lines end

equations where segments end

$$\int_{x=0}^{x=1}\int_{y=0}^{y=2x} x \, dy \, dx$$

vertical lines start

initial point

$x=1$

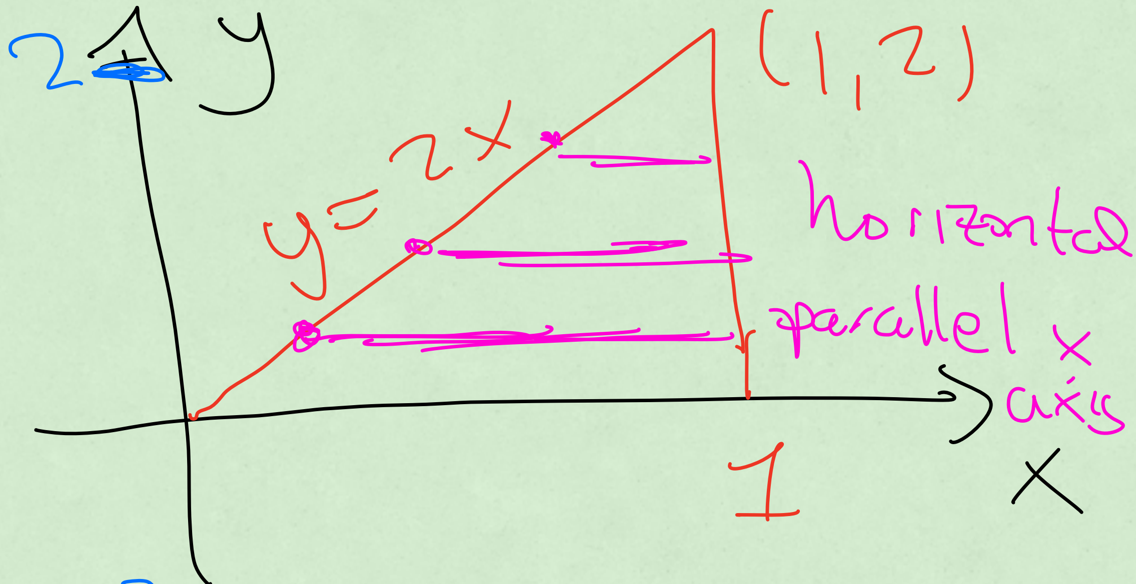
$$= \int_{x=0} \left(xy \right)_{y=0}^{y=2x} dx$$

$$= \int_0^1 (2x^2 - 0)\, dx$$

$$= \left. \frac{2x^3}{3} \right|_{x=0}^{x=1}$$

$$= \boxed{\frac{2}{3}}$$

Horizontal Slice



$$\int_{y=0}^{y=2} \left[\int_{x=\frac{y}{2}}^{x=1} x\,dx \right] dy$$

$$= \int_0^2 \frac{x^2}{2} \Big|_{x=\frac{y}{2}}^{x=1} dy$$

$$= \int_0^2 \left(\frac{1}{2} - \frac{y^2}{8} \right) dy$$

$$= \left(\frac{y}{2} - \frac{y^3}{24} \right) \Bigg|_{y=0}^{y=2}$$

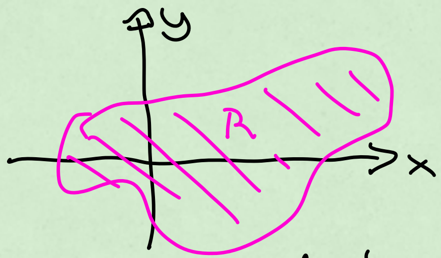
$$= \frac{2}{2} - \frac{8}{24} - (0 - 0)$$

$$= 1 - \frac{1}{3}$$

$$= \boxed{\frac{2}{3}}$$

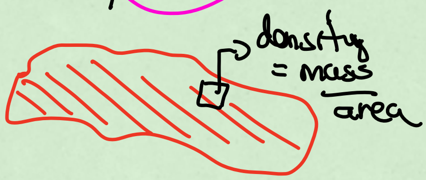
Lecture 16 (15.2, 15.3, 15.4)

Double integrals: $f(x,y)$ and some region R on the xy plane



$$\underbrace{\iint_R f(x,y)\,dA}$$

meaning = "sum" of the quantities $\underbrace{f}_{\text{height box}}\ \underbrace{dA}_{\text{base of the boxes}}$



interpretation of $f(x,y)$	meaning of $\iint_R f(x,y)\,dA$
height above xy plane	= "sum" of height · area = volume
$f(x,y)$ = density = mass per unit area	= "sum" of density · area = sum of $\frac{\text{mass}}{\text{area}}$ · area = mass = total mass of surface
$f(x,y)$ = charge per unit area	= "sum" of density · area = sum of $\frac{\text{charge}}{\text{area}}$ · area = total charge
$f(x,y)$ = the constant 1	= sum of 1 · area = sum of area = area of region R

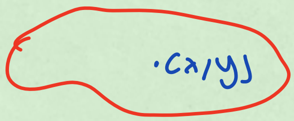
volume = area · height

volume = area · 1

pressure = $\dfrac{\text{force}}{\text{area}}$

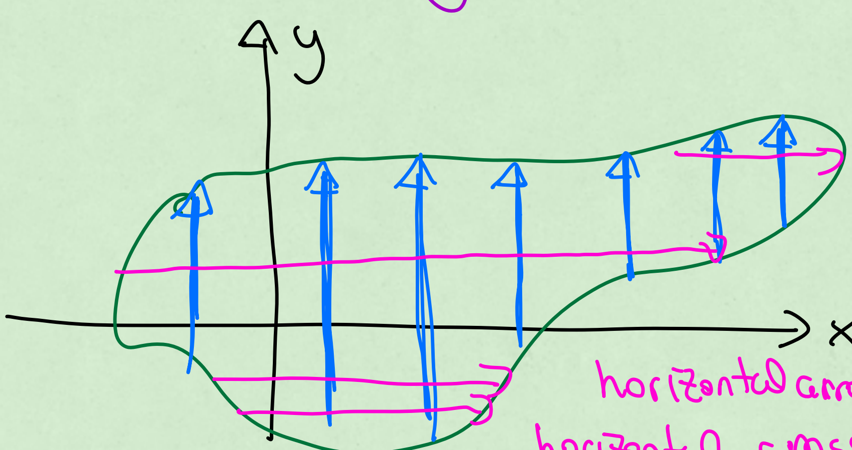
pressure · area = force

f(x,y) = pressure at the point (x,y)



$$\iint_R f(x,y)\,dA = \text{total force exerted on plate}$$

Rules of the game



vertical arrows (south to north)
vertical cross sections
order dy dx

$$\int^{\text{largest value of } x} \left[\int^{\text{curve where vertical arrows end}} f(x,y)\,dy \right] dx$$

horizontal arrows (west to east)
horizontal cross sections
order dx dy

$$\int_{\text{smallest}}^{\text{largest value of } y} \left[\int_{\text{curve}}^{\text{curve where horizontal arrows end}} f(x,y)\,dx \right] dy$$

smallest value of x

bounds are numbers

curve where the vertical arrows start

bounds can depend on x.

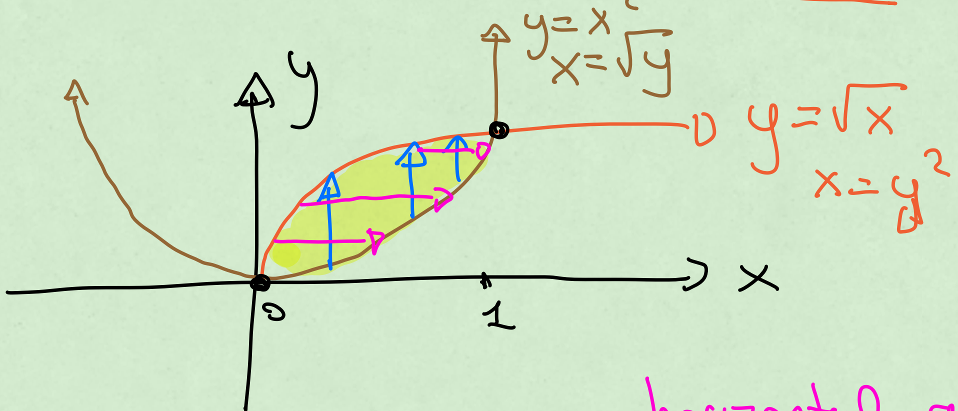
value of y

where horizontal arrows start

bounds can depend on y.

Examples

Region R = region in the first quadrant between the curves $\boxed{y=\sqrt{x}}$ and $\boxed{y=x^2}$



vertical arrows ($dy\,dx$)

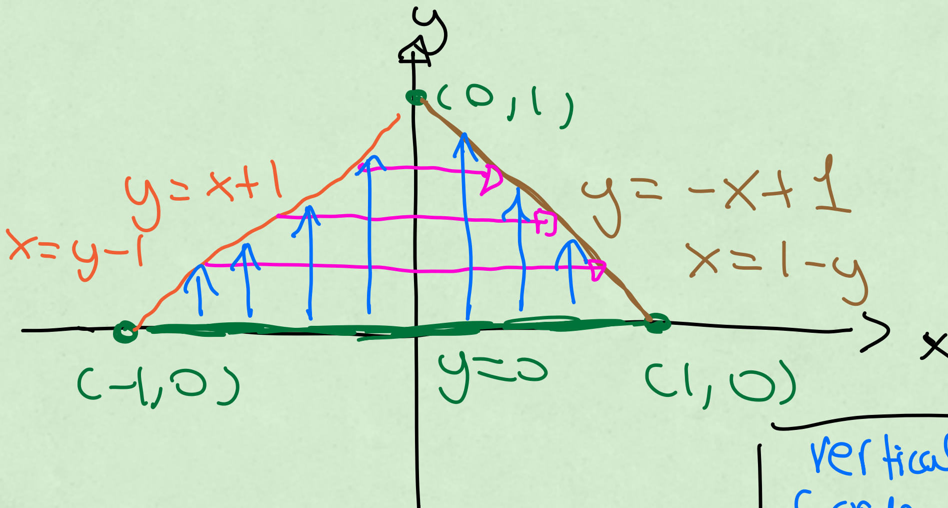
$$\int_0^1 \left[\int_{x^2}^{\sqrt{x}} f(x,y)\,dy\right] dx$$

horizontal arrows ($dx\,dy$)

$$\int_0^1 \left[\int_{y^2}^{\sqrt{y}} f(x,y)\,dx\right] dy$$

Another example

region = inside of a triangle whose vertices are $(-1,0)$, $(0,1)$, $(1,0)$



horizontal arrows (order dxdy)

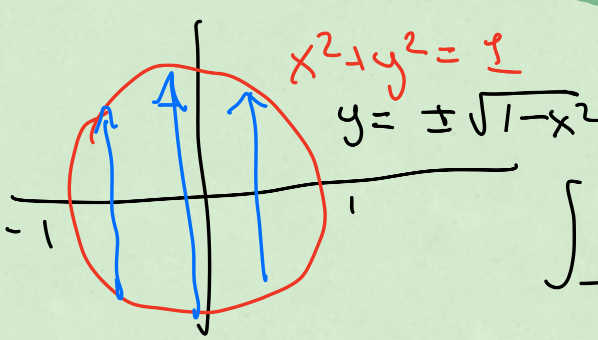
$$\int_0^1 \left[\int_{y-1}^{1-y} f(x,y)\, dx \right] dy$$

vertical arrows (order dy dx)

$$\int_{-1}^{0} \left[\int_0^{x+1} f(x,y)\, dy \right] dx$$

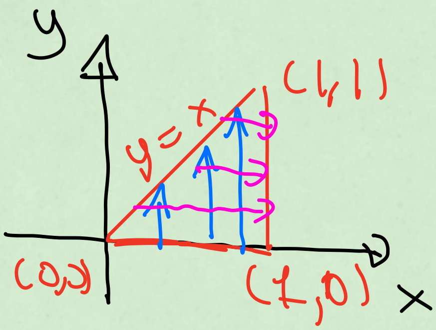
$$+$$

$$\int_0^1 \left[\int_{y=0}^{-x+1} f(x,y)\, dy \right] dx$$



$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\, dy dx$$

example: $f(x,y) = \frac{\sin x}{x}$



$$\int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \frac{\sin x}{x}\, y \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 \frac{\sin x}{x} \cdot x \, dx$$

$$= \int_0^1 \sin x \, dx$$

$$= -\cos x \Big|_0^1$$

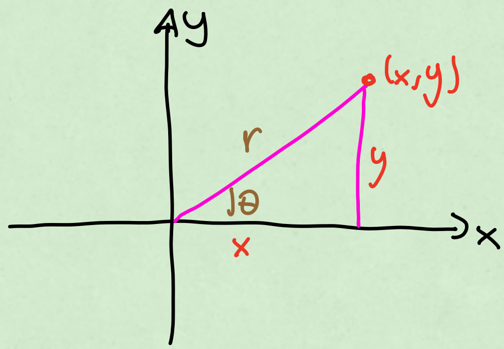
$$= -\cos 1 + \cos 0$$

$$= 1 - \cos 1$$

$$\int_0^1 \int_y^1 \boxed{\frac{\sin x}{x}\, dx}\, dy$$

don't know how to integrate this

Lecture 17 (Polar coordinates 15.4)



$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$ $\left[r^2 = x^2+y^2\right]$

$\tan\theta = \frac{y}{x} \to \theta = \arctan\left(\frac{y}{x}\right)$

$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} \to \theta = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \to \theta = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$

strict rule $0 \le r$
(no negative values for "r")

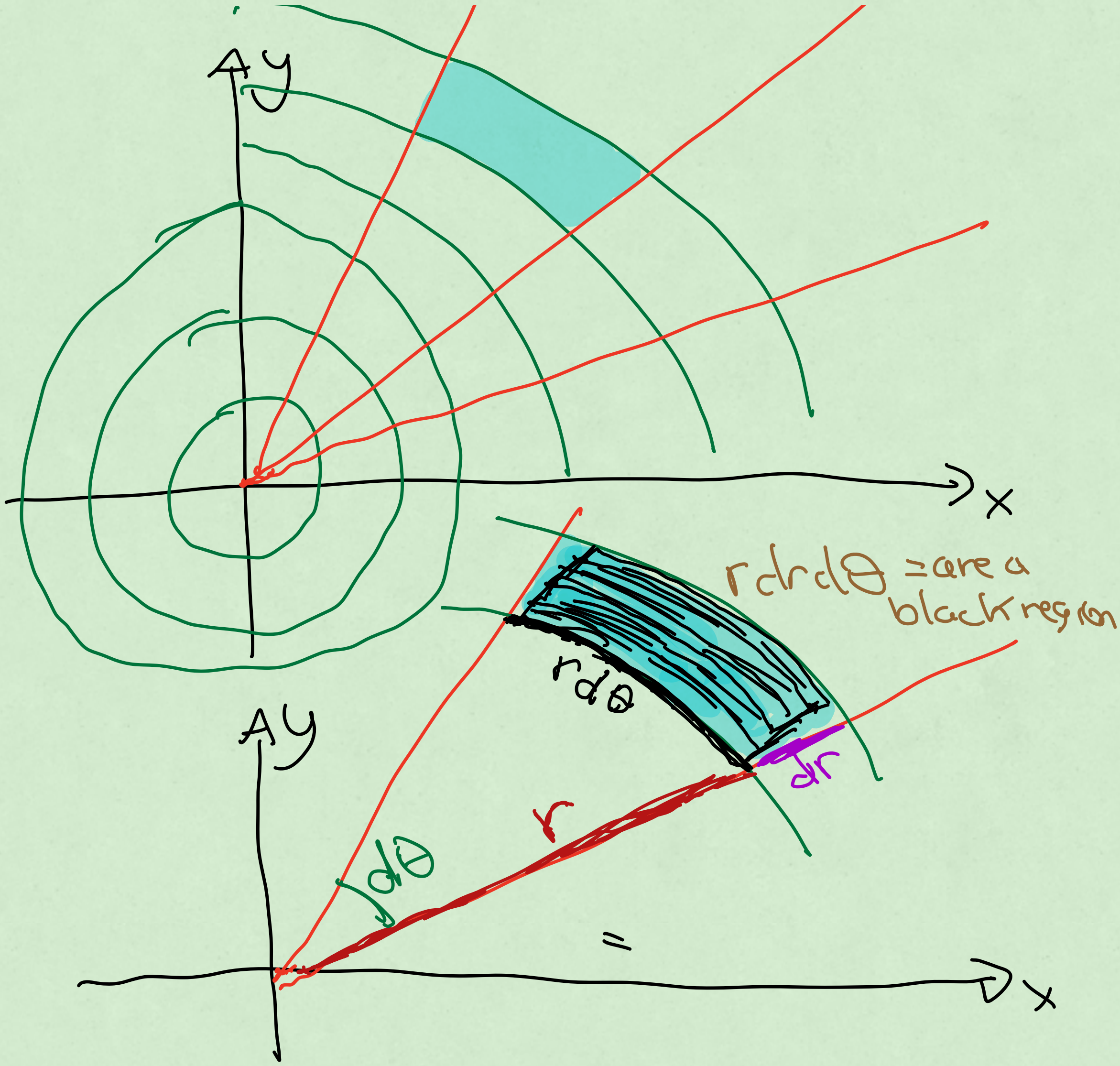
more flexible $0 \le \theta \le 2\pi$

Double integrals in cartesian coordinates

$\int\int f(x,y)\, \boxed{dy\,dx}$ or $\int\int f(x,y)\, \boxed{dx\,dy}$

$\underbrace{dA}_{\text{area of a tiny rectangle}} = dy\,dx = dx\,dy$

Area in polar coordinates?



area in polar coordinates

$$\boxed{dA = r\,d\theta\,dr = r\,dr\,d\theta}$$

so in polar coordinates the integral will look like

$$\iint f\, r\,dr\,d\theta \quad \text{or} \quad \iint f\, r\,d\theta\,dr$$

Analogue of horizontal or vertical arrows

↳ we use radical arrows in polar coordinates

$$\int_{\square}^{\square}\int_{\square}^{\square} f\, r\, \underbrace{dr}\, d\theta$$

Upper bound of outer integral: largest of θ in the region

Lower bound of outer integral: smallest angle of θ in the region

Upper bound of inner integral: equation where radical arrows exit the region (can depend on θ)

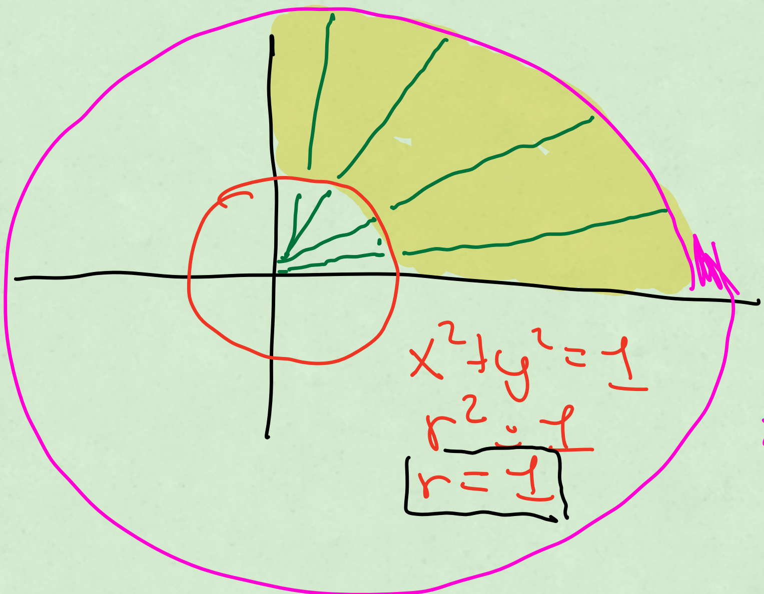
Lower bound of inner integral: equation where radical arrows enter the region (can depend on θ)

example:

set up an integral for $f(x,y) = xy$ in polar coordinates.

Region is = the stuff in the first quadrant between the circles

$x^2+y^2=1$ and $x^2+y^2=9$



$x^2+y^2=1$

$r^2=1$

$\boxed{r=1}$

$x^2+y^2=9$

$r^2=9$

$\boxed{r=3}$

example

$$\boxed{\begin{aligned} f &= xy \\ &= r\cos\theta\, r\sin\theta \\ &= r^2\cos\theta\sin\theta \end{aligned}}_{*}$$

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=1}^{r=3} \underbrace{f}_{*}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_1^3 \underbrace{r^2\cos\theta\sin\theta}_{*}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_1^3 r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^4}{4}\Big|_{r=1}^{r=3} \cos\theta \sin\theta \, d\theta$$

$$= \left(\frac{81}{4} - \frac{1}{4}\right) \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= 20 \cdot \frac{\sin^2\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2}$$

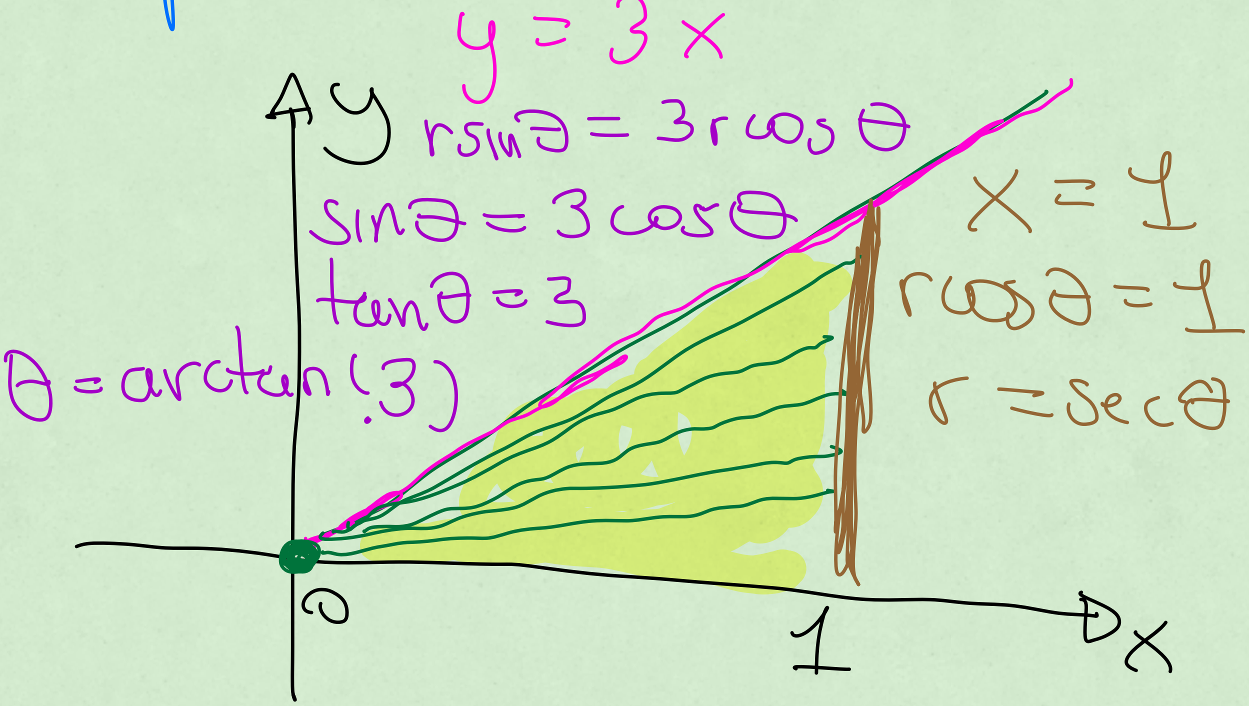
$$= 20 \cdot \frac{1}{2}$$

$$= \boxed{10}$$

Rewrite

$$\int_0^1 \int_0^{3x} \boxed{y} \, \underbrace{\boxed{dy\,dx}}_{dA}$$

as a double integral in polar coordinates



$$\int_{\theta=0}^{\theta=\arctan(3)} \int_{r=0}^{r=\sec\theta} r\sin\theta \; r\,dr\,d\theta$$

Change $\int_0^{\pi/2} \int_0^{\cos\theta} \frac{1}{r}\, dr\, d\theta$

to a double integral in cartesian coordinates

Drawing:

$r = \cos\theta$

$$\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}}$$

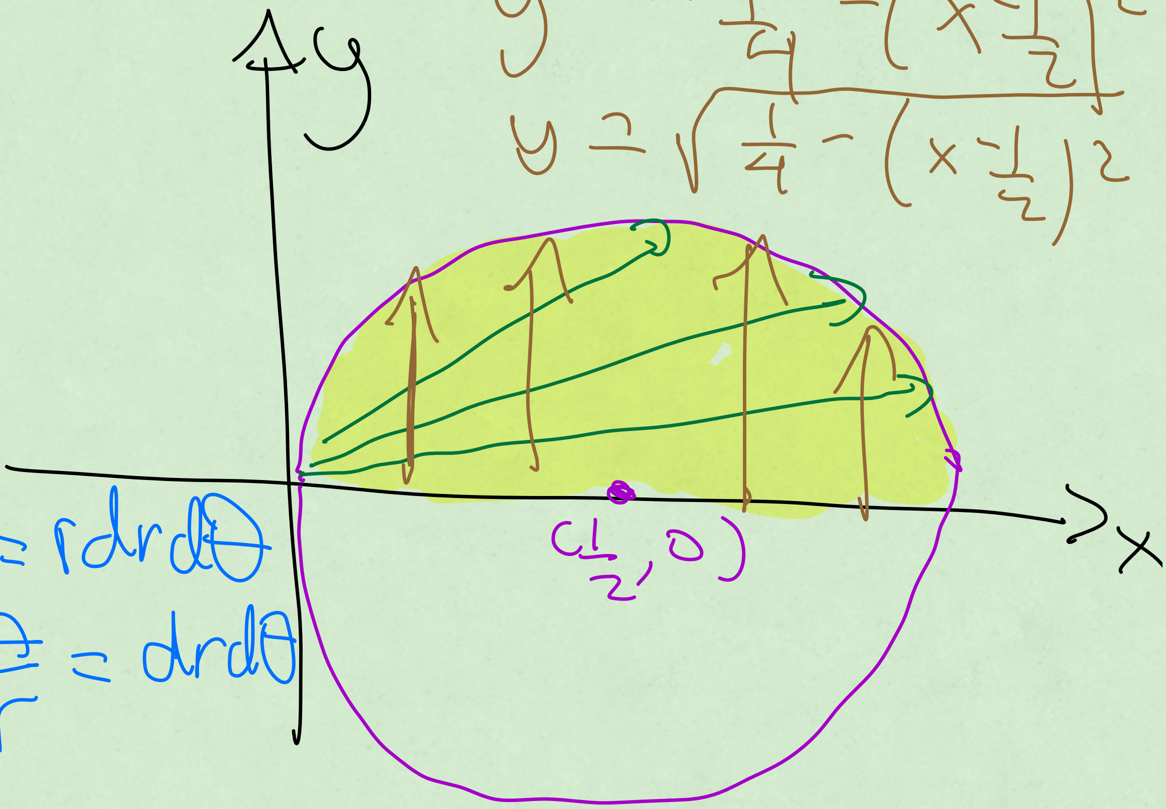
$$x^2+y^2 = x$$

$$x^2 - x + y^2 = 0$$

$$\boxed{\left(x-\frac{1}{2}\right)^2 + y^2 = \frac{1}{4}}$$

$$y^2 = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$$

$$y = \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}$$



$dA = r\,dr\,d\theta$

$\frac{dA}{r} = dr\,d\theta$

$$\int_0^{\pi/2} \int_0^{\cos\theta} \frac{1}{r} \boxed{dr\,d\theta}$$

$dA = dy\,dx = r\,dr\,d\theta$

$$= \int_0^{\pi/2} \int_0^{\cos\theta} \frac{1}{r} \quad \frac{dA}{r}$$

$$= \int_0^{\pi/2} \int_0 \frac{1}{r^2} \, |dA|$$

$$= \int_0^1 \int_0^{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}} \frac{1}{x^2+y^2} \, dy \, dx$$

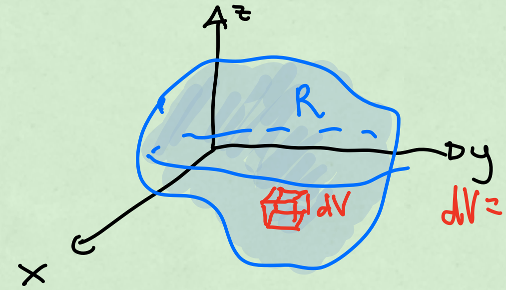
Lecture 18 (15.5)

Triple integrals in cartesian coordinates

$f(x,y,z)$ = function of three variables

$$\iiint_{\boxed{R}} f(x,y,z)\,dV$$

some region of 3d space

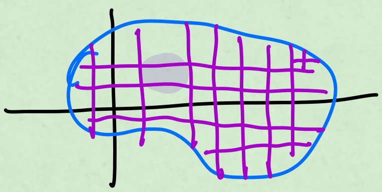


dV = tiny volume

→ sum of quantities " $f(x,y,z)$ · little volumes dV "

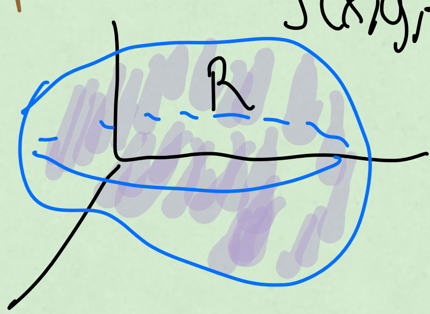
$$dV = \underbrace{dz\,dy\,dx}_{\text{most common}} = dz\,dx\,dy = dy\,dz\,dx = dy\,dx\,dz$$
$$= dx\,dy\,dz = dx\,dz\,dy$$

most common order of integration so we will start with this one



units of $\iiint f(x,y,z)\,dV$ = units of f · units volume

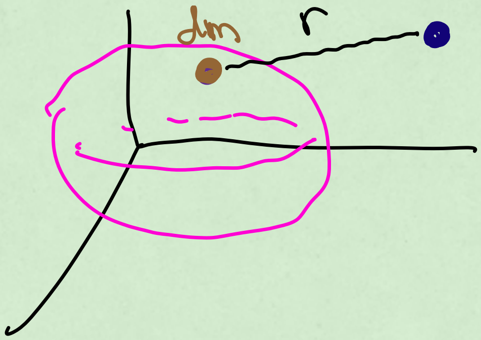
examples



$f(x,y,z)$ = mass per unit volume = density

$$\underbrace{\iiint_R f\,dV = \frac{\text{mass}}{\text{volume}} \cdot \text{volume} = \text{mass}}_{\text{total mass of planet}}$$

M



$$dF = \frac{GM(dm)}{r^2}$$

total force

$$= \iiint dF$$

ρ = density function

$$= \iiint \frac{GM\,dm}{r^2}$$

$$\frac{dm}{dV} = \rho$$

$$= \iiint \frac{GM\rho}{r^2}\,dV$$

Rules for finding Bounds

$$\underbrace{\int\left[\int\right.}_{\text{shadow or projection this region makes on the xy plane}}\left[\int_{\boxed{}}^{\text{eq surface arrows exit (in terms of x and y)}} f(x,y,z)\,dz\right]dy\Big]\,dx$$

arrows parallel to z axis

[eq surface arrows exit (in terms of x and y)]

equation surface where arrows enter (in terms of x, y)

shadow or projection this region makes on the xy plane and then you find the bounds of the shadow as a double integral

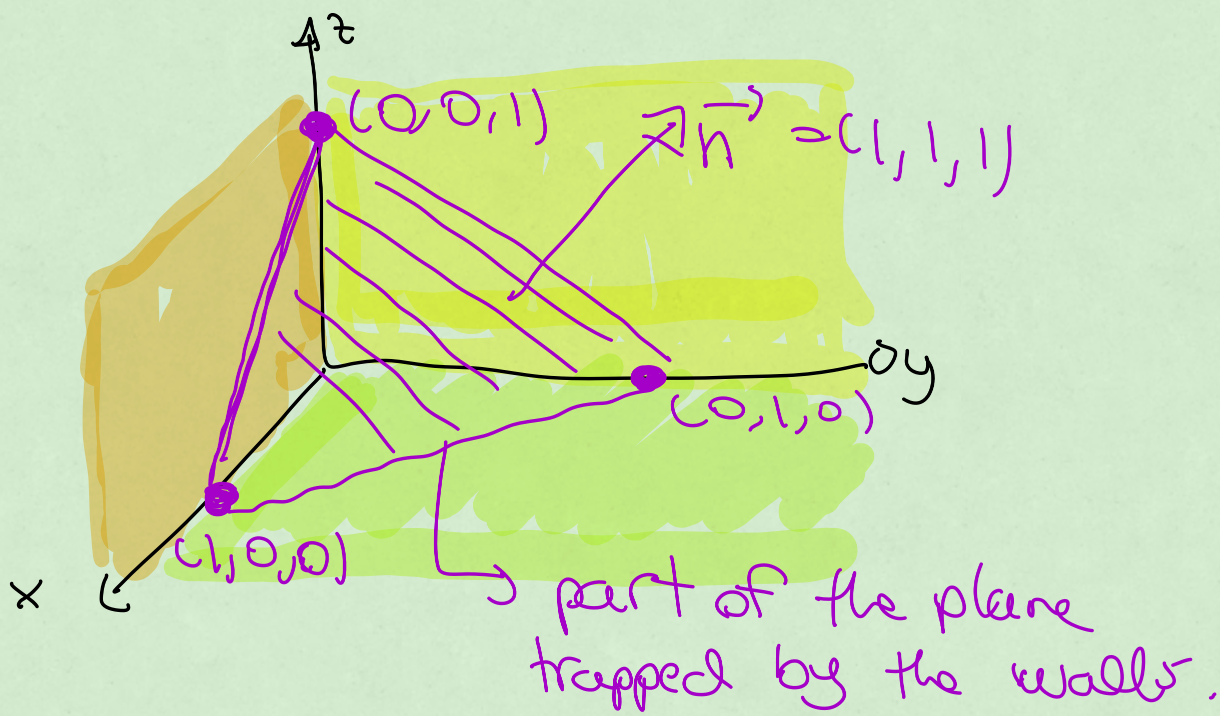
Find the bounds for the triple integral

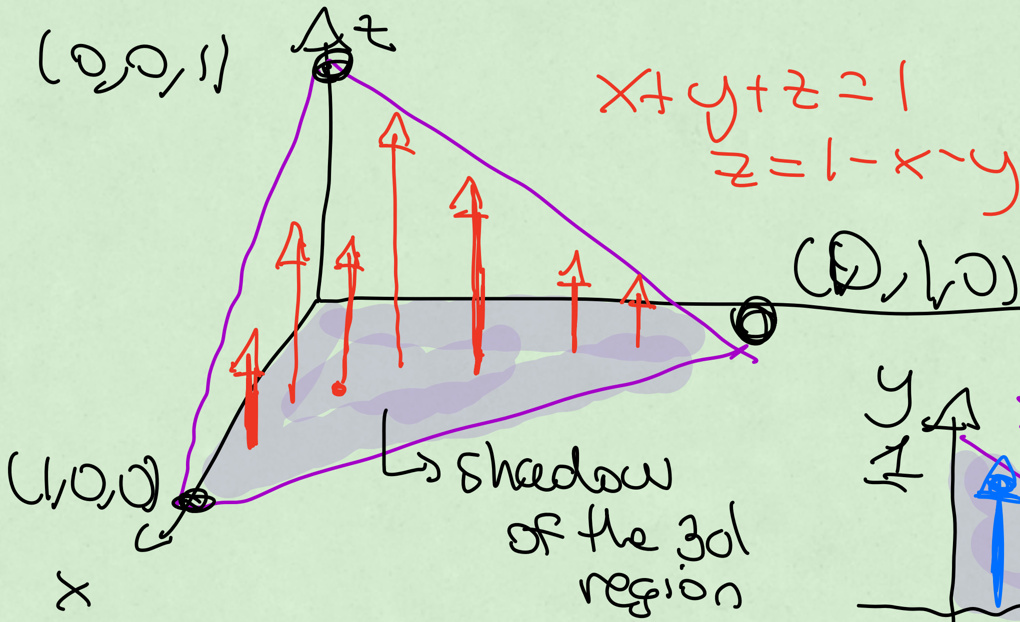
$$\iiint_R (5x - 3y)\, z \quad dz\,dy\,dx$$

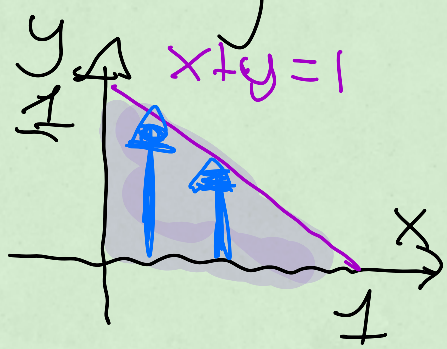
where the region R is the tetrahedron (pyramid) determined by the planes

$x = 0$, $y = 0$, $z = 0$

$\boxed{x + y + z = 1}$ normal vector $\vec{n} = (1, 1, 1)$



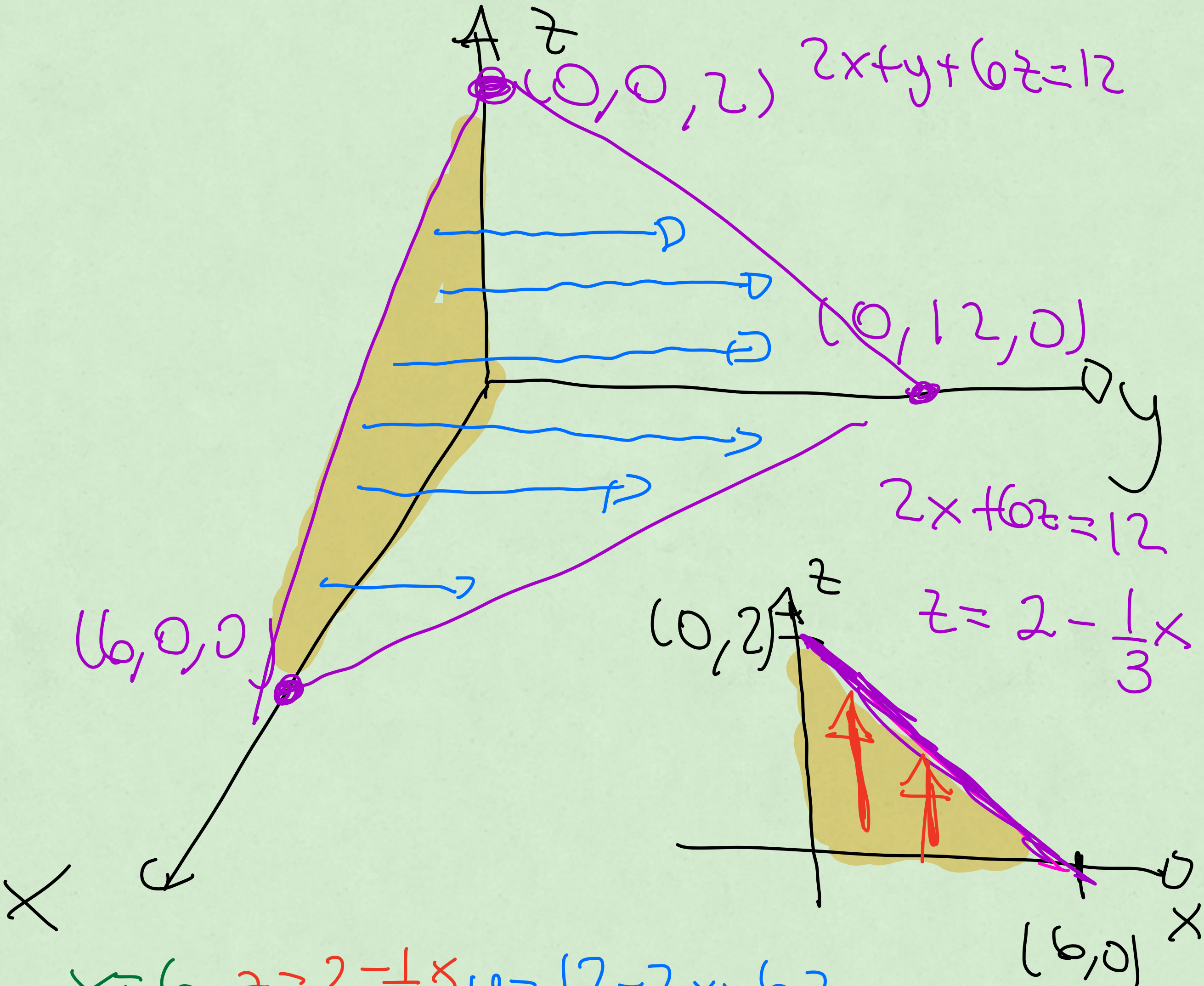




$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y}(5x-3y)\,z\;dz\,dy\,dx$$

same problem but now with plane $2x+y+6z=12$

and order $dy\,dz\,dx$



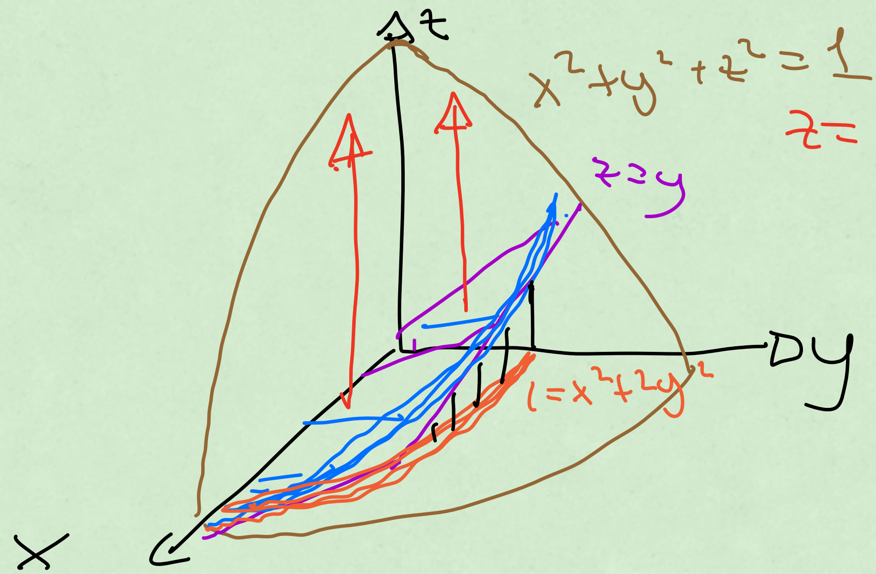
$z = 2 - \frac{1}{3}x$

$$\int_{x=0}^{x=6} \int_{z=0}^{z=2-\frac{1}{3}x} \int_{y=0}^{y=12-2x-6z} (5x-3y)z \, dy \, dz \, dx$$

Bounds for

$$\iiint_R (x+z)\, dz\, dy\, dx$$

R = region in the first octant $(x \geq 0, y \geq 0, z \geq 0)$ which is inside the sphere $x^2+y^2+z^2=1$ and above the plane $\boxed{z=y}$



intersection plane sphere

$$\sqrt{1-x^2-y^2}=y$$

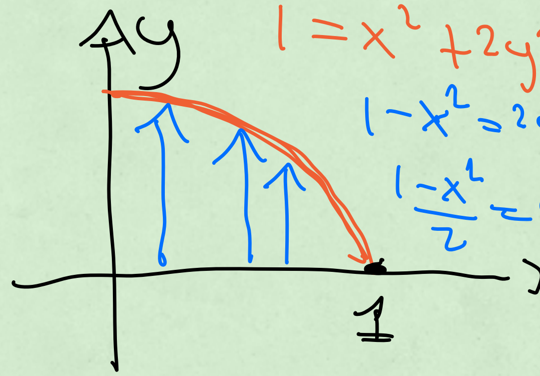
$$1-x^2-y^2=y^2$$

$$\boxed{1=x^2+2y^2}$$

$y=\sqrt{1-x^2}$

$$\int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{\frac{1-x^2}{2}}} \int_{z=y}^{z=\sqrt{1-x^2-y^2}} (x+z)\, dz\, dy\, dx$$

ellipse



$1 = x^2 + 2y^2$

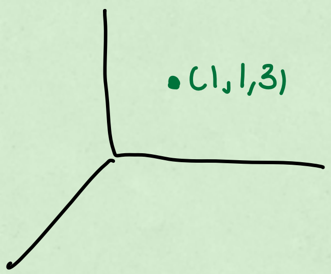
$1 - x^2 = 2y^2$

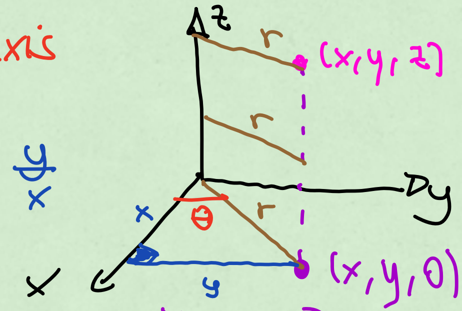
$\frac{1-x^2}{2} = y^2$

Lecture 19 (cylindrical / spherical coordinates)

Cylindrical coordinates
= polar coordinates + z axis

$\tan\theta = \frac{y}{x}$





$r = \sqrt{x^2+y^2}$
= distance of a point to the z axis

$x = r\cos\theta$
$y = r\sin\theta$
z remains the same variable
[not being changed in terms of new variables]

$x^2+y^2 = 4 \xrightarrow{\text{polar}} \boxed{r=2}$

$x^2+y^2+z^2 = 4 \xrightarrow{\text{cylindrical}} r^2+z^2 = 4$

$$\boxed{dV = dz\,dy\,dx = dz\,dA = dz\,r\,dr\,d\theta = r\,dz\,dr\,d\theta}$$

cylindrical coordinates $\boxed{dV = r\,dz\,dr\,d\theta}$

Triple integral in cylindrical coordinates

arrows parallel to z axis

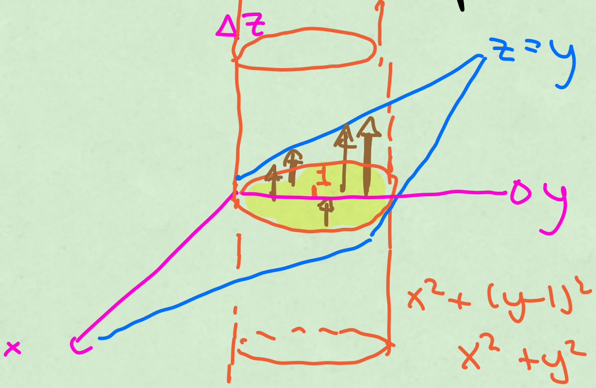
eq where arrows exit [in terms of r, θ]

$$\underbrace{\int\int}_{\text{look at the shadow on the xy plane, but now the bounds are written in polar coordinates}} \left[\int_{\text{equation where arrows enter}}^{\text{eq where arrows exit}} f\; r\,dz\right] dr\,d\theta$$

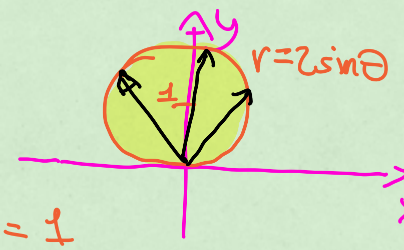
equation where arrows enter can be in terms of r, θ

look at the shadow on the xy plane, but now the bounds are written in polar coordinates example: find the volume of the region which is:

- above the xy plane.
- inside the cylinder $x^2+(y-1)^2=1$
- below the plane $z=y$.



$\boxed{z=r\sin\theta}$



$$x^2+(y-1)^2=1$$
$$x^2+y^2-2y+1=1$$
$$x^2+y^2-2y=0$$
$$x^2+y^2=2y$$
$$r^2=2r\sin\theta$$
$$\boxed{r=2\sin\theta}$$

$$\text{volume} = \iiint dV$$

$$= \int_0^{\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=r\sin\theta} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} rz\Big|_{z=0}^{z=r\sin\theta} dr\,d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} r^2 \sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{r^3}{3}\Big|_{r=0}^{r=2\sin\theta} \cdot \sin\theta\,d\theta$$

$$= \int_0^{\pi} \frac{8}{3} \sin^3\theta \cdot \sin\theta \, d\theta$$

$$= \frac{8}{3} \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= \frac{8}{3} \int_0^{\pi} [\sin^2\theta]^2 \, d\theta$$

$$= \frac{8}{3} \int_0^{\pi} \left(\frac{1-\cos(2\theta)}{2} \right)^2 d\theta$$

$$= \frac{8}{3} \underbrace{\int_0^{\pi} 1 - \frac{2\cos(2\theta) + \cos^2(2\theta)}{4} \, d\theta}_{\frac{3\pi}{8}}$$

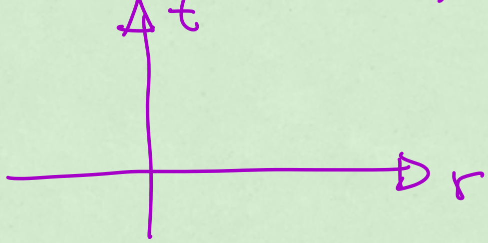
(wolfram alpha !!)

$$= \boxed{\pi}$$

cylindrical coordinates when there is symmetry:

↳ θ does not show up in any of the equations that you are given to determine the bounds.

in this case you can find the bounds by drawing a 2d picture, where the vertical axis is "z", horizontal axis is "r"

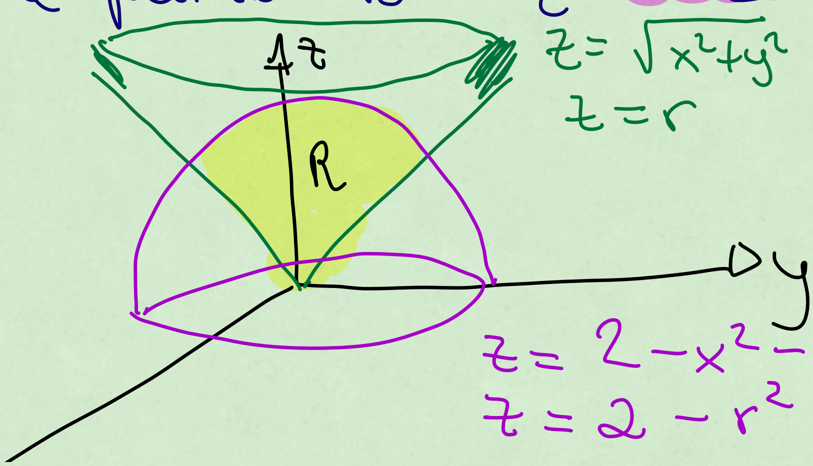


[rz plane]

Example: find the integral in cylindrical coordinates of

$$\iiint_R x \, dz \, dy \, dx$$

where R is the region above the cone $z = \sqrt{x^2+y^2}$ and below the paraboloid $z = 2 - x^2 - y^2$



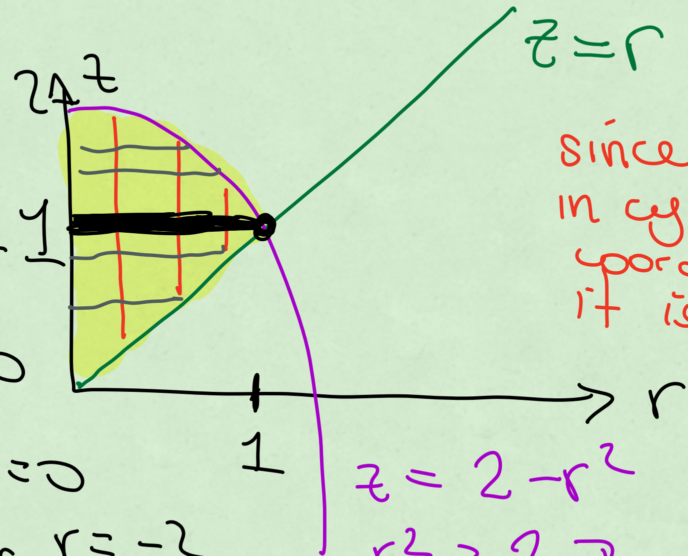
$z = \sqrt{x^2+y^2}$

$z = r$

$z = 2 - x^2 - y^2$

$z = 2 - r^2$

× ↙ neither equation has θ so can draw them on the rz plane



$r = 2 - r^2$

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$\boxed{r = 1}$ or $r = -2$

$z = 2 - r^2$

$r^2 = 2 - z$

$r = \sqrt{2-z}$

since $r \geq 0$ in cylindrical coordinates it is enough to draw only the stuff on the positive r axis.

$$\iiint x \, \overbrace{dz\,dy\,dx}^{dV}$$

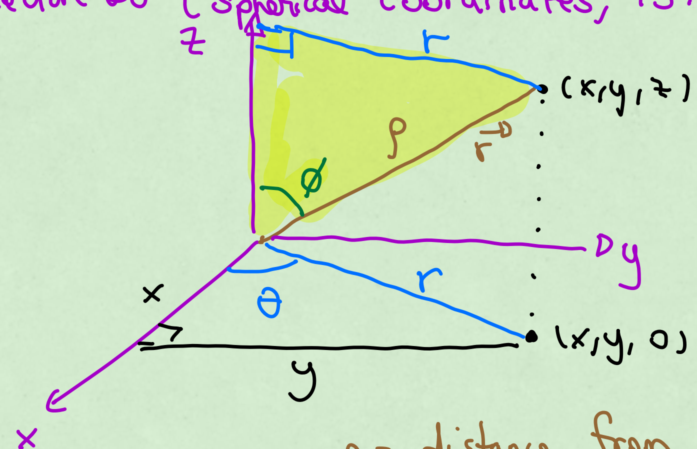
$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r\cos\theta \; r\,dz\,dr\,d\theta$$

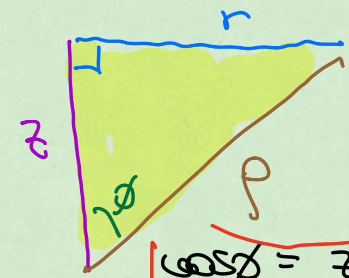
$$= \int_0^{2\pi} \int_0^{1} \int_{r=0}^{r=z} r\cos\theta \; r\,dr\,dz\,d\theta$$

$$\int^{2\pi} \int^{2} \int^{r=\sqrt{2-z}}$$

$$+\int_{0}\int_{1}\int_{r=0} r\cos\theta \; r\,dr\,dz\,d\theta$$

Lecture 20 (spherical coordinates, 15.7)





$$\boxed{\cos\phi = \frac{z}{\rho}}$$

$$\boxed{\sin\phi = \frac{r}{\rho}}$$

ρ = distance from point to the origin $(0,0,0)$

$$\rho = \sqrt{x^2+y^2+z^2} = |\vec{r}|$$

θ = angle from polar coordinates $0 \le \theta < 2\pi$

ϕ = angle between z axis and the position vector $\vec{r}$

$0 \le \phi \le \pi$, $\phi = 0$ (north pole)

$\phi = \pi/2$ (equator, xy plane)

$\phi = \pi$ (south pole)

$$\boxed{z = \rho\cos\phi} \qquad \boxed{r = \rho\sin\phi}$$

Dictionary

cartesian		cylindrical		spherical
x	$=$	$r\cos\theta$	$=$	$\rho\sin\phi\cos\theta$
y	$=$	$r\sin\theta$	$=$	$\rho\sin\phi\sin\theta$
z	$=$	z	$=$	$\rho\cos\phi$

$$dV = dz\,dy\,dx = r\,dz\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

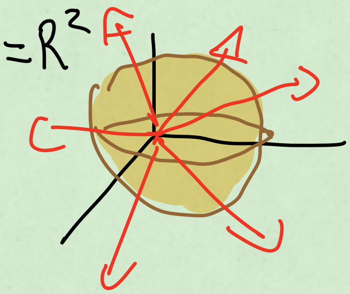
$$\boxed{\begin{array}{c}\rho^2 = x^2+y^2+z^2 \\ r^2 = x^2+y^2\end{array}} \quad \boxed{\rho^2 = r^2+z^2}$$

example: volume of a sphere of radius R

$x^2+y^2+z^2=R^2$

$\rho^2 = R^2$

$\boxed{\rho = R}$



volume

$$= \iiint dV$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi}\int_{\rho=0}^{\rho=R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

draw light rays emanating from the origin in all directions

$x^2+y^2+z^2$ Aside

$= \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2\cos^2\phi$

$= \rho^2\left[\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right) + \cos^2\phi\right]$

$$= \rho^2 (\sin^2\phi + \cos^2\phi)$$

$$= \rho^2$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{\rho^3}{3}\Big|_0^{\rho=R} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{R^3}{3} \sin\phi \, d\phi \, d\theta$$

$$= \frac{R^3}{3} \int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi} d\theta$$

$$= \frac{2R^3}{3} \int_0^{2\pi} d\theta$$

$$= \boxed{\frac{4\pi R^3}{3}}$$

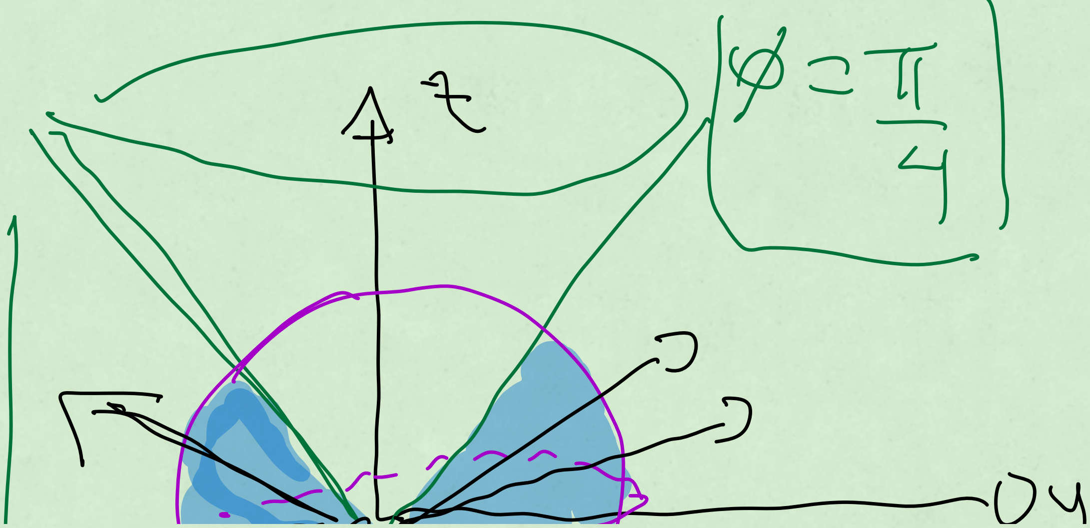
Example:

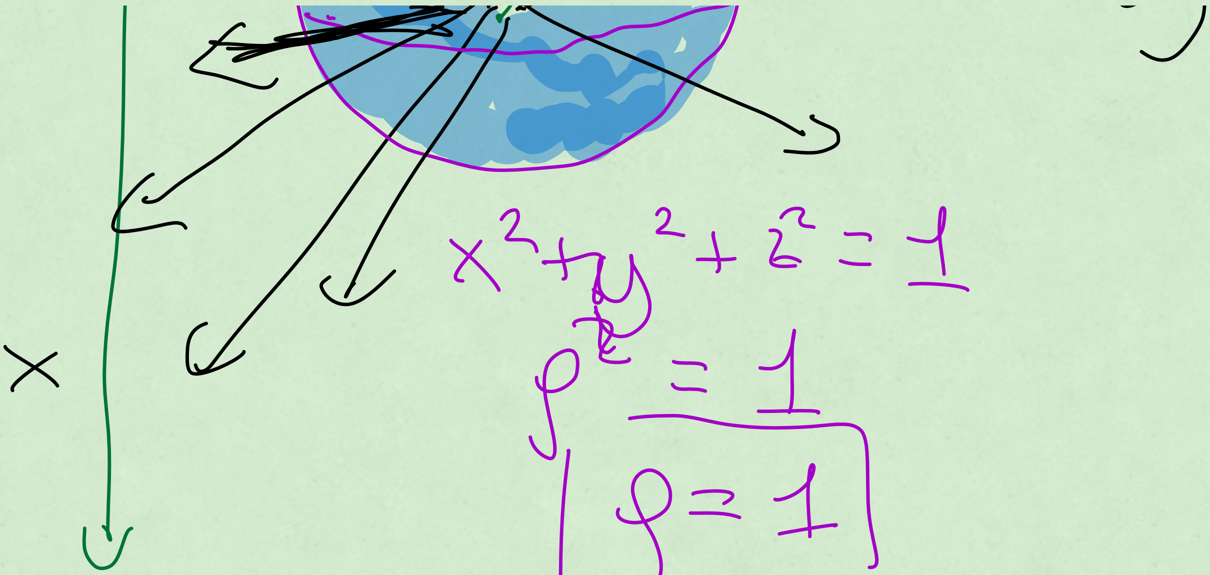
cone $\boxed{z = \sqrt{x^2+y^2}}$

sphere $\boxed{x^2+y^2+z^2 = 1}$

Find an integral the gives you the volume of the region inside the sphere but below the cone.

$\boxed{\phi = \frac{\pi}{4}}$





$$x^2+y^2+z^2=1$$

$$\rho^2 = 1$$

$$\boxed{\rho = 1}$$

$$z=\sqrt{x^2+y^2}$$

$$\rho\cos\phi =$$

$$\rho\cos\phi = \rho\sin\phi$$

$$1 = \tan\phi$$

$$\boxed{\phi = \frac{\pi}{4}}$$

$\theta = 2\pi \quad \phi = \pi \quad \rho = 1$

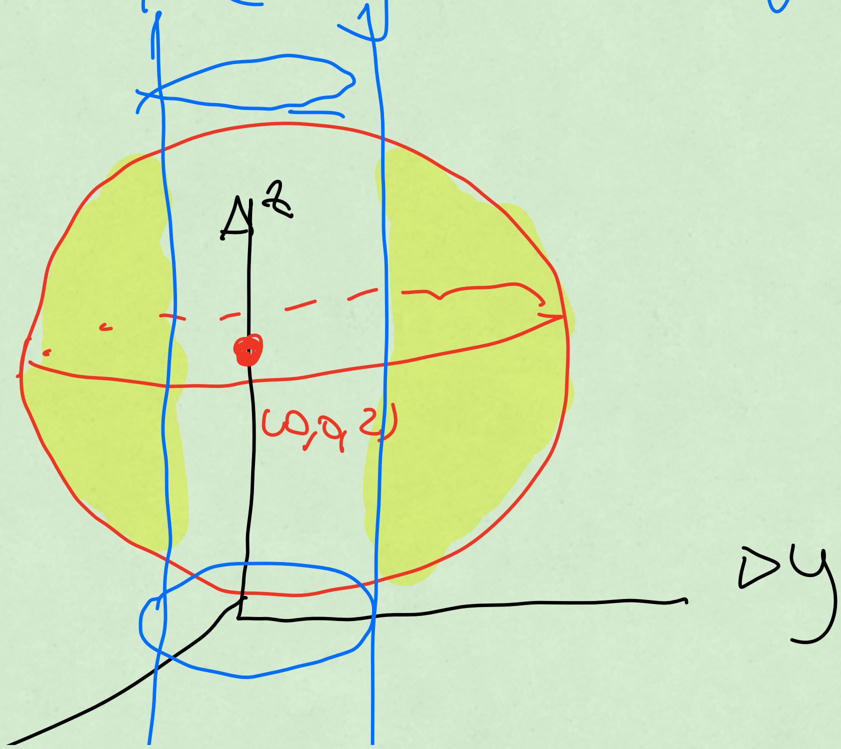
$$\int_{\theta=0} \int_{\phi=\frac{\pi}{4}} \int_{\rho=0} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

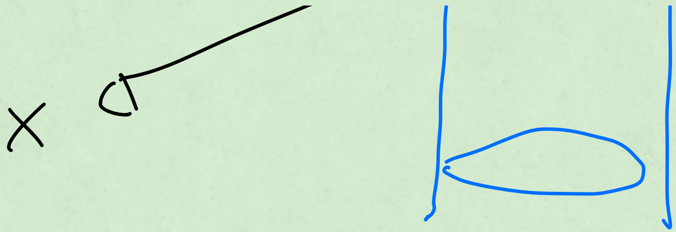
volume of the region

inside the sphere

$$x^2+y^2+(z-2)^2=4$$

outside the cylinder $x^2+y^2=1$

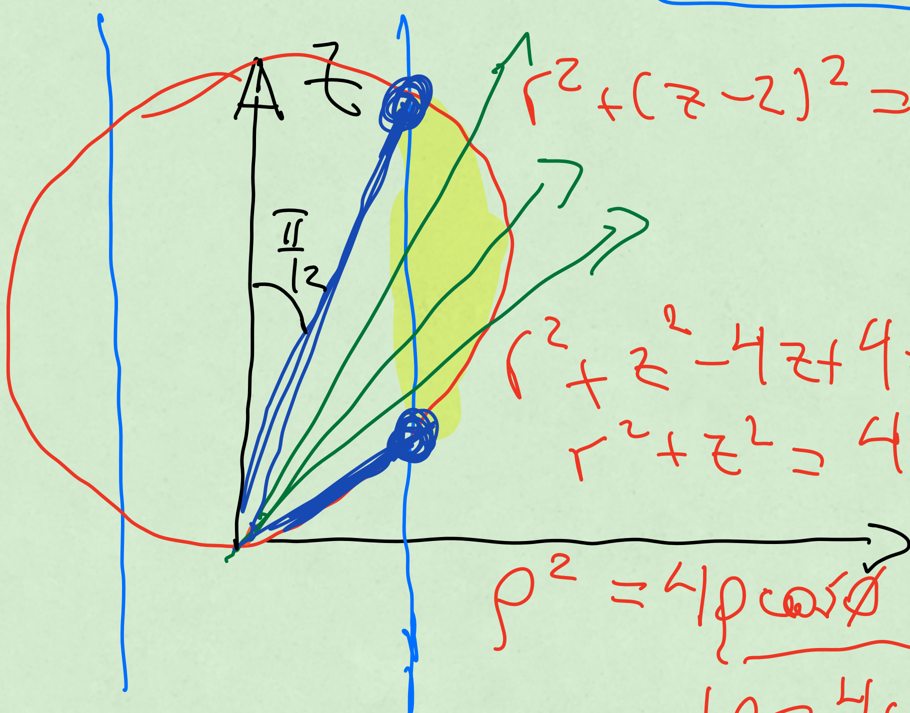




Trick: write the equations first in cylindrical before finding everything in spherical.

$$x^2+y^2+(z-2)^2=4 \qquad x^2+y^2=1$$

$$r^2+(z-2)^2=4 \qquad \boxed{r=1}$$



$$r^2+(z-2)^2=4$$

$$r^2+z^2-4z+4=4$$

$$r^2+z^2=4z$$

$$\rho^2=4\rho\cos\phi$$

$$\rho=4\cos\phi$$

$r = 1$

$\rho \sin\phi = 1$

$\rho = \csc\phi$

Trick

the bounds for spherical can be found from the rz plane by drawing radial arrows

$$\int_0^{2\pi} \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \int_{\rho = \csc\phi}^{\rho = 4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

angles

$$4\cos\phi = \csc\phi$$

$$4\cos\phi = \frac{1}{\sin\phi}$$

$$4\cos\phi\sin\phi = 1$$

$$2\sin(2\phi) = 1$$

$$\sin(2\phi) = \frac{1}{2}$$

$$2\phi = \frac{\pi}{6} \quad \text{or} \quad 2\phi = \pi - \frac{\pi}{6}$$

$$\phi = \frac{\pi}{12} \qquad \phi = \frac{\pi}{2} - \frac{\pi}{12}$$

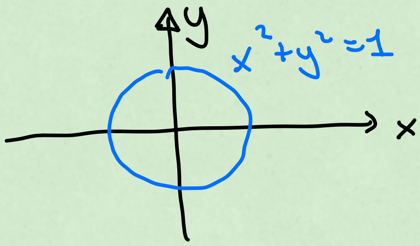
$$\phi = \frac{5\pi}{12}$$

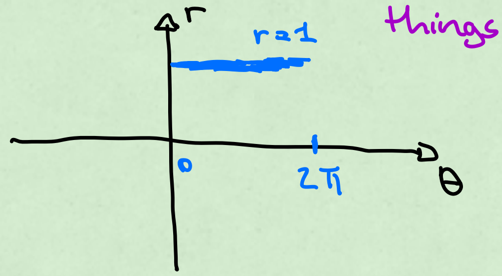
Lecture 21 (15.8)

$x^2+y^2=1$

$r=1$

$$\left.\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array}\right\}$$ change of variables that turns some curves like circles into less interesting things





conceptual assignment

$\boxed{y=\frac{e^2}{x}}$

$x=ve^{-u}$

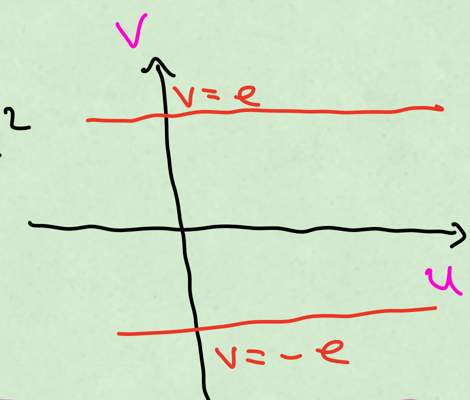
$y=ve^{u}$

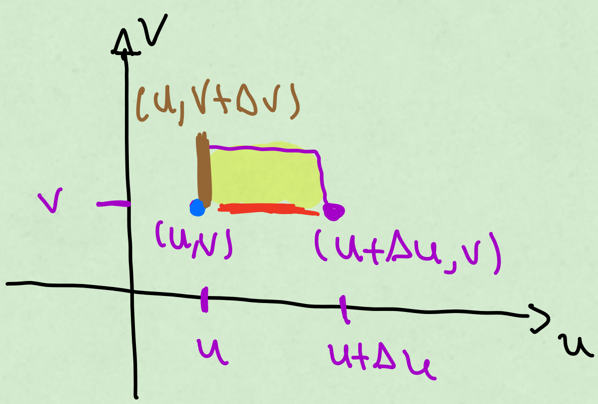
$xy=e^2$

$ve^{-u}\cdot ve^{u}=e^2$

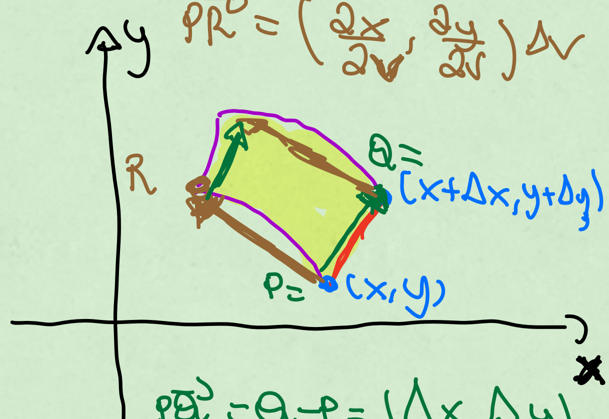
$v^2=e^2$

$\boxed{v=\pm e}$





$\vec{PR}=\left(\frac{\partial x}{\partial v},\frac{\partial y}{\partial v}\right)\Delta v$



$\vec{PQ}=Q-P=(\Delta x,\Delta y)$

$=\left(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u}\right)\Delta u$

$\Delta x=\frac{\Delta x}{\Delta u}\,\Delta u\approx\frac{\partial x}{\partial u}\,\Delta u$

$\Delta y=\frac{\Delta y}{\Delta u}\,\Delta u\approx\frac{\partial y}{\partial u}\,\Delta u$

area parallelogram $= |\vec{PQ} \times \vec{PR}|$

$$= \left| \left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, 0\right) \times \left(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v, 0\right) \right|$$

$$= \left| \left(0, 0, \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\Delta v \Delta u\right) \right|$$

$$= \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| \Delta v \Delta u$$

Jacobian and change of variables

If you make a change of variables

$x = x(u, v)$, $y = y(u, v)$

Jacobian matrix $= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$

Jacobian $= J =$ det of Jacobian matrix

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of variables formula

$$\iint f(x,y)\,\boxed{dy\,dx}$$

$$= \iint \underbrace{f(u,v)}_{\text{write function in terms of } u,v}\ \boxed{\underbrace{|J|}_{\text{absolute value of the Jacobian}}\ dv\,du}$$

example (secretely polar coordinates)

$x = v\cos(u) \qquad y = v\sin(u)$

Jacobian matrix $= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$

$= \begin{pmatrix} -v\sin(u) & v\cos(u) \\ \cos(u) & \sin(u) \end{pmatrix}$

$J = \det \text{matrix} = -v\sin^2 u - v\cos^2 u = -v$

change of variables $\qquad |J| = |-v| = v$

$$\iint f(x,y)\,dy\,dx = \iint f\ |J|\,dv\,du$$

$$= \iint f\ v\,dv\,du$$

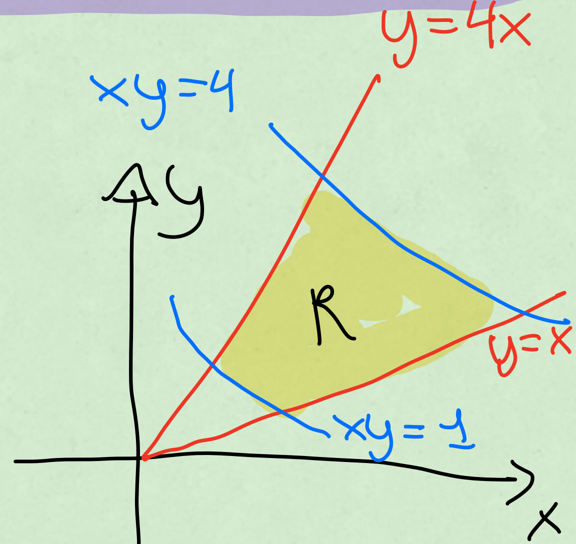
$v = r$
$u = \theta$

$$= \iint f\ r\,dr\,d\theta$$

---

example : integrate

$$\iint_R (x^2+y^2)\,dy\,dx$$



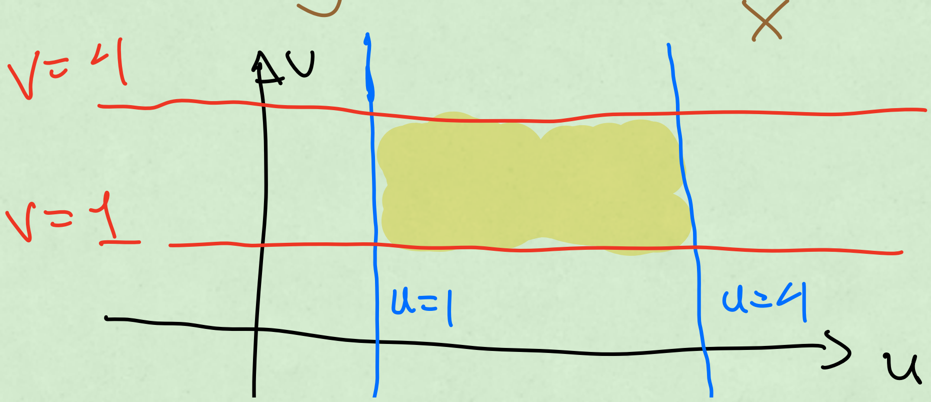
change of variables

$u = xy \quad , \quad v = \dfrac{y}{x}$

$y = x \qquad\qquad y = 4x$

$\dfrac{y}{x} = 1 \qquad\qquad \dfrac{y}{x} = 4$

$v = 1 \qquad\qquad v = 4$



need x, y in terms of u, v

$u = xy$ $\qquad$ $v = \frac{y}{x}$

$\frac{u}{x} = y \longrightarrow v = \frac{u}{x} \cdot \frac{1}{x}$

$x^2 = \frac{u}{v}$

$x = \sqrt{\frac{u}{v}} = u^{1/2} v^{-1/2}$

$y = \frac{u}{u^{1/2} v^{-1/2}} = u^{1/2} v^{1/2}$

$$\boxed{\begin{aligned} x &= u^{1/2} v^{-1/2} \\ y &= u^{1/2} v^{1/2} \end{aligned}}$$

Jacobian matrix

$$= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} u^{-1/2} v^{-1/2} & \frac{1}{2} u^{-1/2} v^{1/2} \\ -\frac{1}{2} u^{1/2} v^{-3/2} & \frac{1}{2} u^{1/2} v^{-1/2} \end{pmatrix}$$

$J = \det = \frac{1}{4} v^{-1} + \frac{1}{4} v^{-1} = \frac{1}{2v}$

$$\iint \boxed{(x^2+y^2)}\,\boxed{dy\,dx} = \int_1^4 \int_1^4 (uv^{-1}+uv)\,\underbrace{\frac{1}{2v}\,dv\,du}_{|J|\,dv\,du}$$

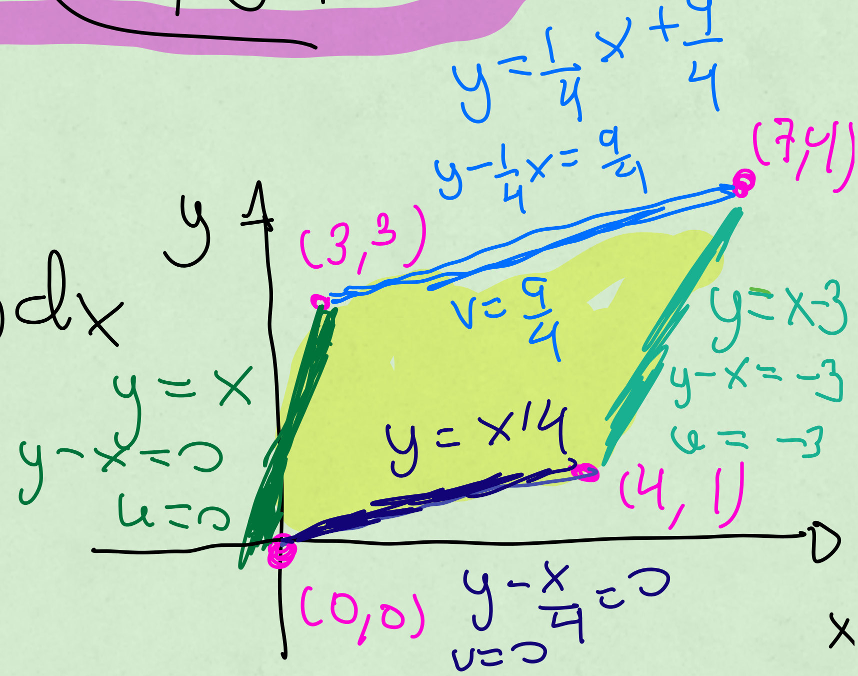
$$= \frac{1}{2}\int_1^4\int_1^4 \left(\frac{u}{v^2}+u\right)dv\,du$$

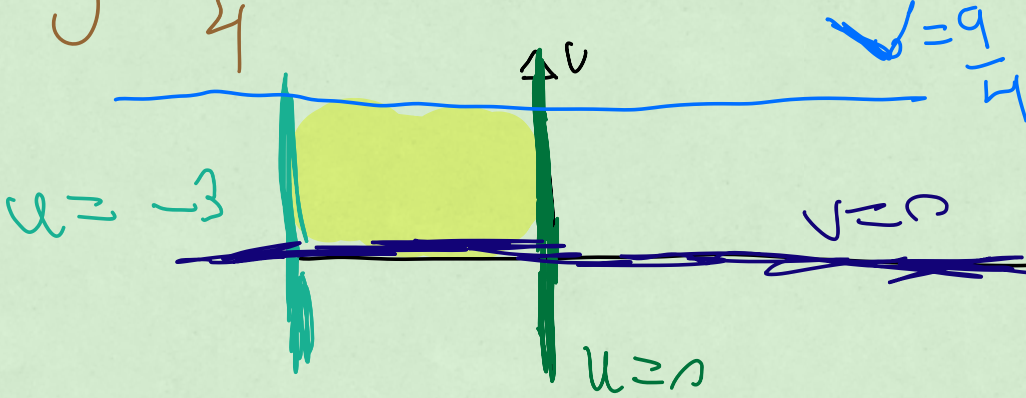
$$= \boxed{\frac{225}{16}}$$

$$\iint e^{4x-y}\,dy\,dx$$

R

$$\begin{cases} u = y - x \\ v = y - \dfrac{x}{4} \end{cases}$$





Find x, y in terms of u, v

(1) $u = y - x$

(2) $v = y - \frac{x}{4}$

(2) − (1)

$v - u = -\frac{x}{4} + x$

$v - u = \frac{3x}{4}$

$$\boxed{\begin{aligned} x &= \frac{4(v-u)}{3} \\ y &= \frac{1}{3}(4v-u) \end{aligned}}$$

Jac matrix

$$= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} & -\frac{1}{3} \\ \frac{4}{3} & \frac{4}{3} \end{pmatrix}$$

$$\det = J = -\frac{16}{9} + \frac{4}{9} = -\frac{4}{3}$$

$$\iint e^{4x-y}\,dy\,dx$$

$$= \int_{-3}^{0} \int_{0}^{9/4} e^{16\left(\frac{v-u}{3}\right) - \frac{1}{3}(4v-u)} \, |J| \, dv\,du$$

$$= \int_{-3}^{0} \int_{0}^{9/4} e^{4v} e^{-5u} \, \frac{4}{3} \, dv\,du$$

$$= \frac{1}{15} \left(e^{15}-1\right)\left(e^{9}-1\right)$$