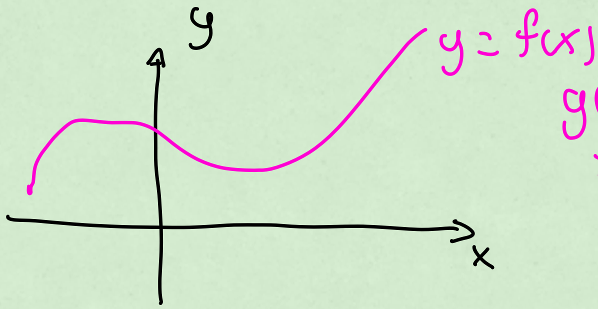
Lecture 8 (14.1 - 14.2)

Functions of many variables and limits

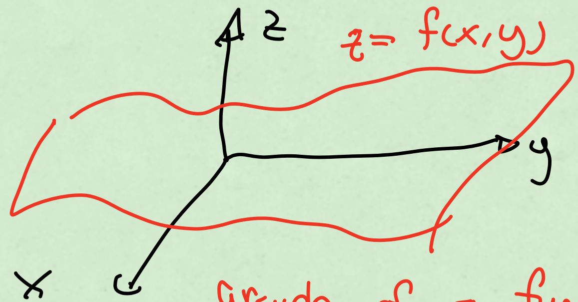


graph of a function of one variable = curve

$y=x^2$

$y=x+1$

$y=\sin(x)$



$z=x^2+y$

$z=\sin(x+y)$

$z=e^{x-y}$

graph of a function of two variables is a surface

Domain of a function $f(x,y)$:
consists of the points (x,y) where the formula (equation) for $f(x,y)$ makes sense.

$$z=f(x,y)=\frac{1}{x-y}$$

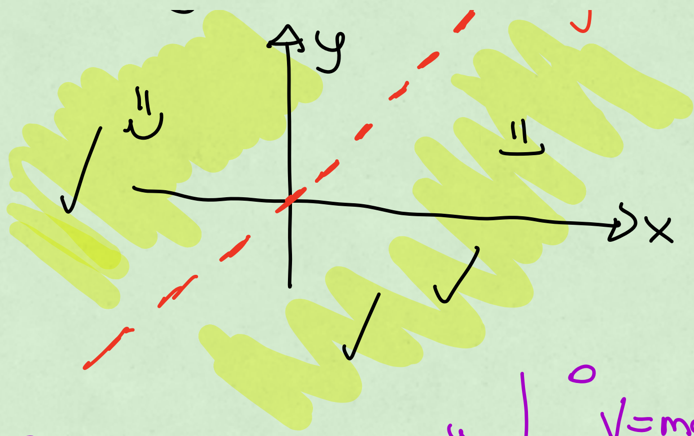
potential issue: denominator $=0$

$$x-y=0$$

$$\boxed{y=x}$$

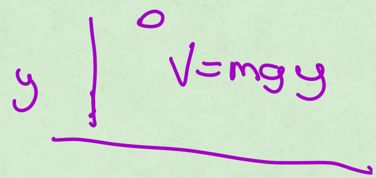
domain of $\frac{1}{x-y}=z$ is any point (x,y) where $x\neq y$.

$\cancel{x=y}$!!



gravitational potential

$V(x,y,z) = \frac{GM}{\sqrt{x^2+y^2+z^2}}$



$f(x,y) = \sqrt{1-x^2-y^2}$

domain?

potential problem: stuff inside square root is negative

we need $\underbrace{1-x^2-y^2 \geq 0}_{\text{where the graph will make sense}}$

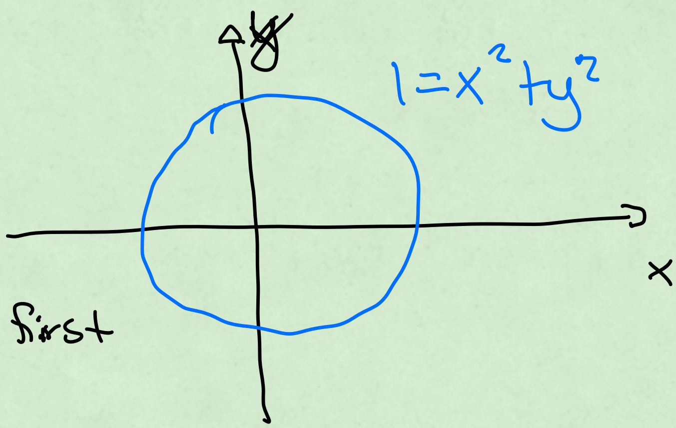
$1-x^2-y^2 \geq 0$

$1 \geq x^2+y^3$

method for drawing inequalities

(1) draw the equality first

$1 = x^2+y^2$

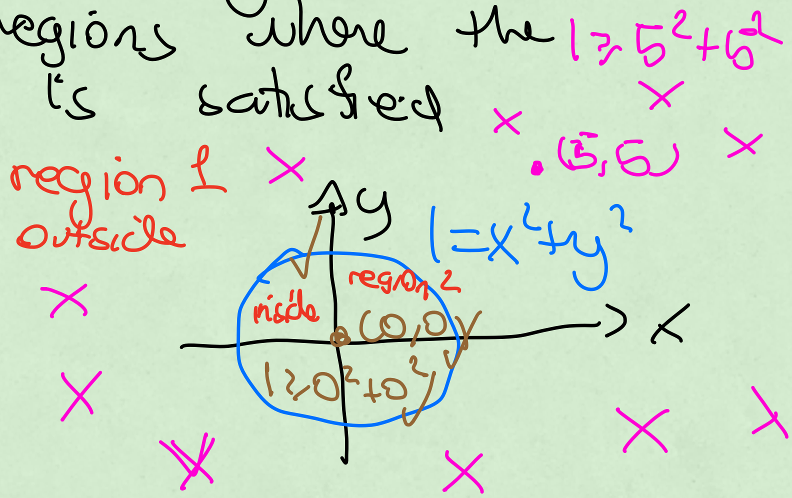


(2) after you do this the plane gets divided into regions and you choose by trial and error the regions where the inequality is satisfied

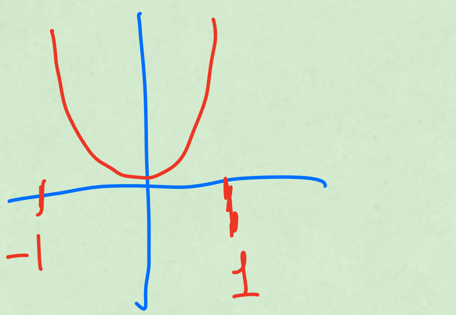
$\underbrace{1 \geq x^2+y^2}$

not satisfied on region 1

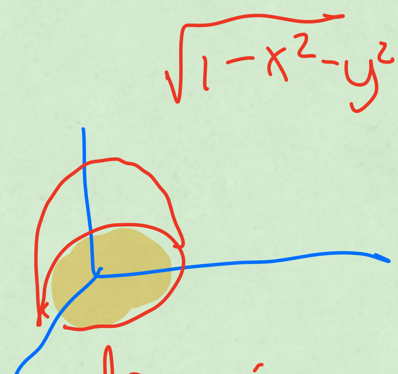
satisfied on region 2



domain of $\sqrt{1-x^2-y^2}$ is the unit disc



domain $[-1,1]$

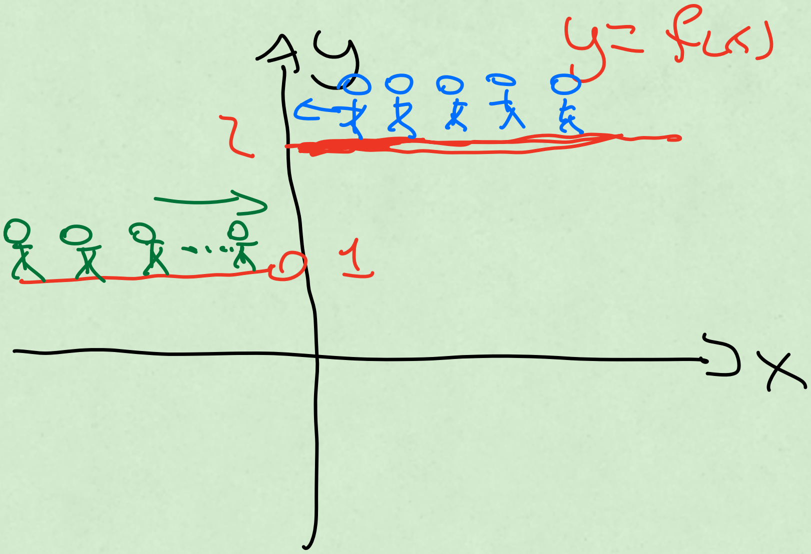


domain is unit disc

Domain of $f(x)$: something on the number line

Domain of $f(x,y)$: some region on the xy plane

Limits



$\lim_{x \to 0} f(x)$ = Does not exist (DNE)

$\lim_{x \to 0^+} f(x) = 2$

$\lim_{x \to 0^-} f(x) = 1$

so no limit since left and right limits

our case

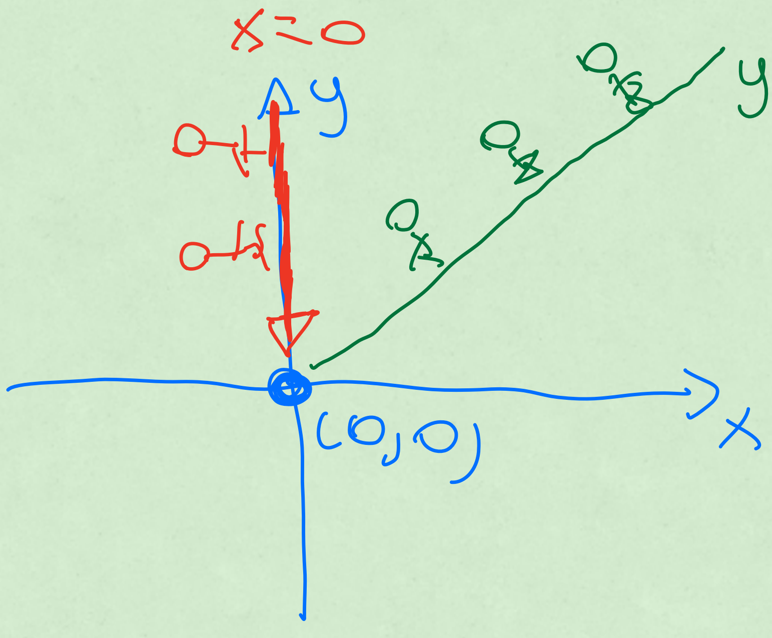
$$\lim_{(x,y)\to(a,b)} f(x,y)$$

now there are a lot more one sided limits than just left or right

Key idea: if you find two "one sided limits" that disagree, then the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ can't exist.

example:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$



limit along the y-axis!

$$\lim_{y\to 0} \frac{2\cdot 0\cdot y}{0^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

limit along the x=y line:

$$\lim_{x\to 0} \frac{2xx}{x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1$$

so limit does not exist since you found two different one sided limits

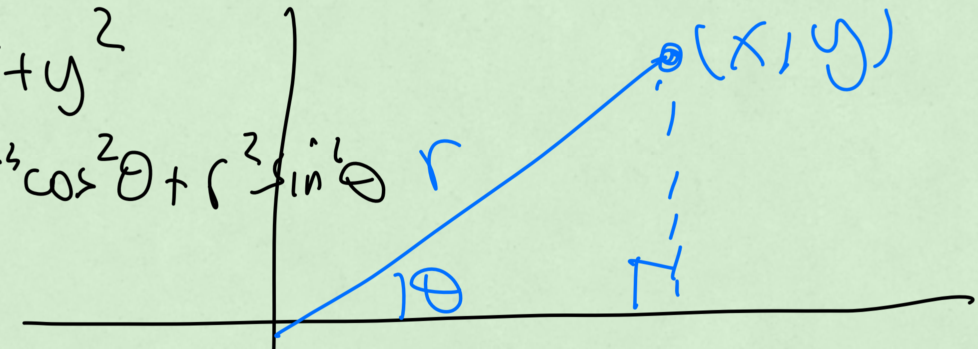
this strategy only works for showing a limit <u>does</u> <u>not</u> exist

For this class, we can use squeeze theorem to show a limit exists

this is done in combination with polar coordinates

x^2+y^2

$= r^2\cos^2\theta + r^2\sin^2\theta$



$x = r\cos\theta$

$y = r\sin\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

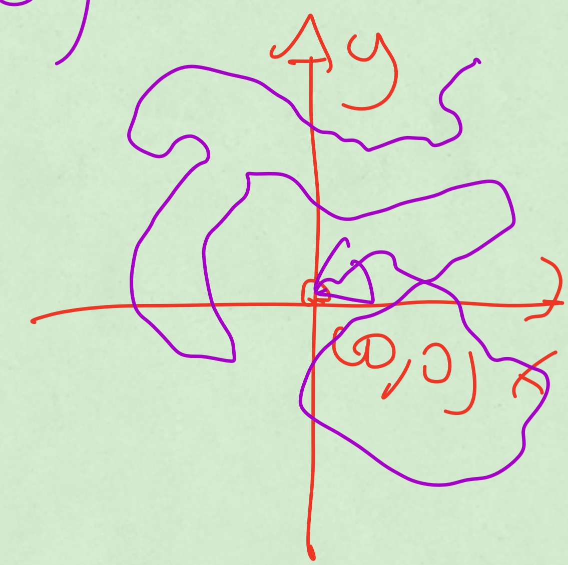
$= r^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$$

$$\lim_{r \to 0} \frac{r^2 \cos^2\theta \; r\sin\theta}{r^2}$$

$$= \lim_{r \to 0} \left(r \cos^2\theta \sin\theta \right)$$

$$= 0$$



squeeze:

$$-1 \le \cos^2\theta \sin\theta \le 1$$

$$-r \le r\cos^2\theta \sin\theta \le r$$

$$0 \qquad\qquad 0 \qquad\qquad 0$$

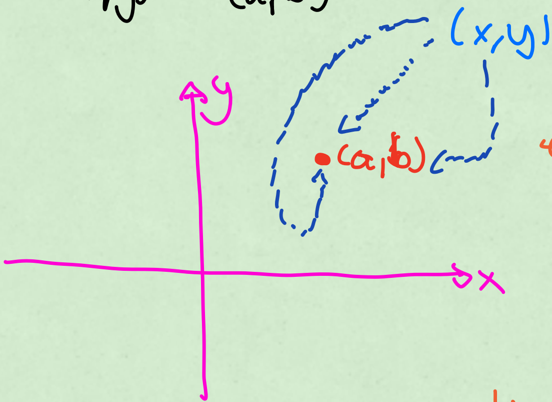
as $r \to 0$ by squeeze as $r \to 0$

Lecture 9 (14.1, 14.2)

Back to limits (14.2)

want to study

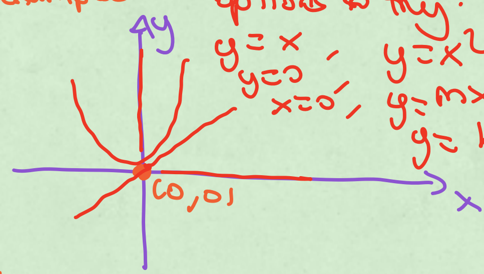
$$\lim_{(x,y)\to(a,b)} f(x,y)$$



Basic strategies

① To show that a limit does not exist, it is enough to find two paths (or trajectories) approaching (a,b) where the limits along these paths give different values

examples



options to try: $y=x$, $y=x^2$, $y=0$, $x=0$, $y=mx$, $y=kx^2$

$$\lim_{(x,y)\to(0,0)}$$

② To show that a limit exists, use an argument like the squeeze theorem.

Rewrite x as $x=r\cos\theta$

y as $y=r\sin\theta$

Change $f(x,y)$ into an expression of r,θ

so if your expression has $\sin\theta$, $\cos\theta$ or combinations of these, then you use the fact they are quantities between -1 and 1.

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example of case 1

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{y-x}$$

$y=0$

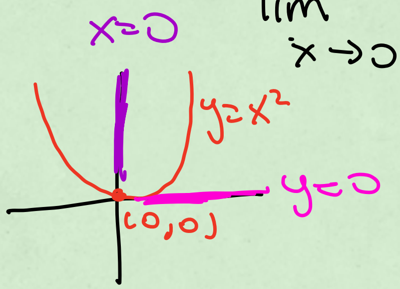
$$\lim_{x\to 0} \frac{x+0}{0-x} = \lim_{x\to 0} \frac{x}{-x} = -1$$

$x=0$

$$\lim_{y\to 0} \frac{0+y}{y-0} = \lim_{y\to 0} \frac{y}{y} = 1$$

different so does no exist

$y=x^2$



$$\lim_{x\to 0} \frac{x+x^2}{x^2-x} = \lim_{x\to 0} \frac{x(1+x)}{x(x-1)}$$

$$= \lim_{x\to 0} \frac{1+x}{x-1} = -1$$

example case 2 $\boxed{\sin^2\theta + \cos^2\theta = 1}$

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$$

$x = r\cos\theta$ $y = r\sin\theta$

$$\lim_{r\to 0} \frac{r^4\cos^4\theta + r^4\sin^4\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$\lim_{r\to 0} \frac{r^4(\cos^4\theta + \sin^4\theta)}{r^2}$$

$$\lim_{r \to 0} r^2 (\cos^4\theta + \sin^4\theta)$$

squeeze argument

$$0 \le \sin^4\theta \le 1$$

$$0 \le \cos^4\theta \le 1$$

so

$$0 \le \sin^4\theta + \cos^4\theta \le \underbrace{2}$$

only need this to be a fixed number so 20, 200, 10 would work

multiply by r^2

$$0 \le \underbrace{r^2(\sin^4\theta + \cos^4\theta)}_{\text{forces this to be 0 by squeeze.}} \le \boxed{2r^2}$$

$0 \downarrow$ already zero

$2r^2 \downarrow$ goes to 0 as $r \to 0$

easier case:
when the function is continuous at the point (a,b)

$f(x,y)$ is continuous at (a,b) if

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}$$

which functions are continuous

- polynomials of x, y
- trigonometric functions which depend on x, y
- exponential functions which depend on x, y

↳ combinations of these all of them are continuos at any point of the domain.

example

$$\lim_{(x,y)\to\left(\frac{\pi}{2},3\right)} \frac{e^{x+y}(y+3)}{\sin(x)} = \frac{e^{\frac{\pi}{2}+3}(3+3)}{\sin\left(\frac{\pi}{2}\right)}$$

$$= 6e^{\frac{\pi}{2}+3}$$

$$\lim_{(x,y)\to(0,2)} \frac{x^2+xy}{x}$$

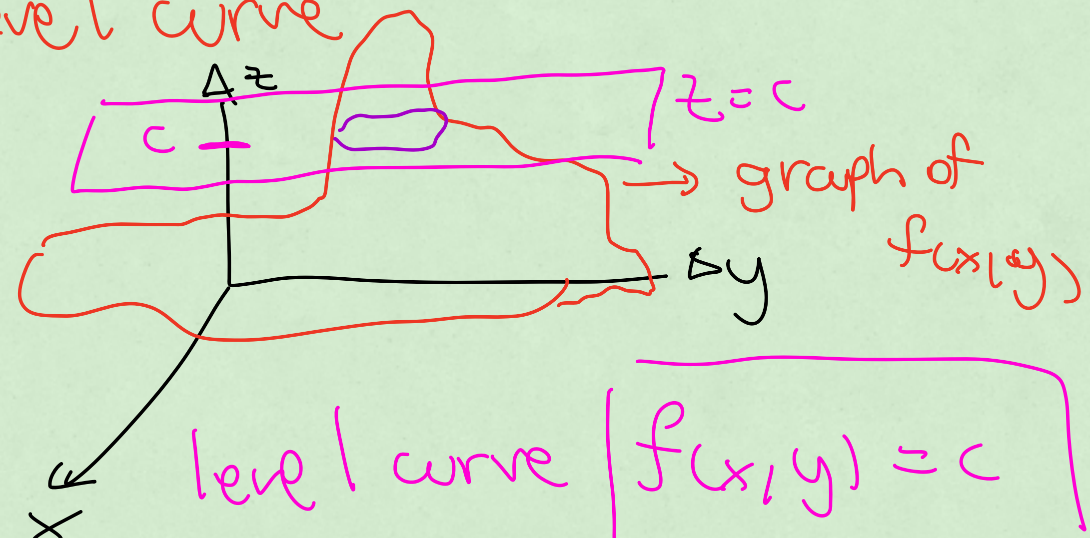
$$= \lim_{(x,y)\to(0,2)} \frac{x(x+y)}{x}$$

$$= \lim_{(x,y)\to(0,2)} (x+y) = 2$$

Back to 14.1

(Level curves and level surface)

Level curve



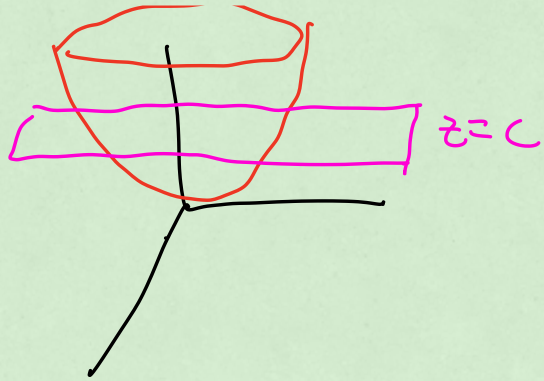
level curve $\boxed{f(x,y) = c}$

consists of all the points (x,y) where the value of "f" equal "c", so where

$$f(x,y) = c$$

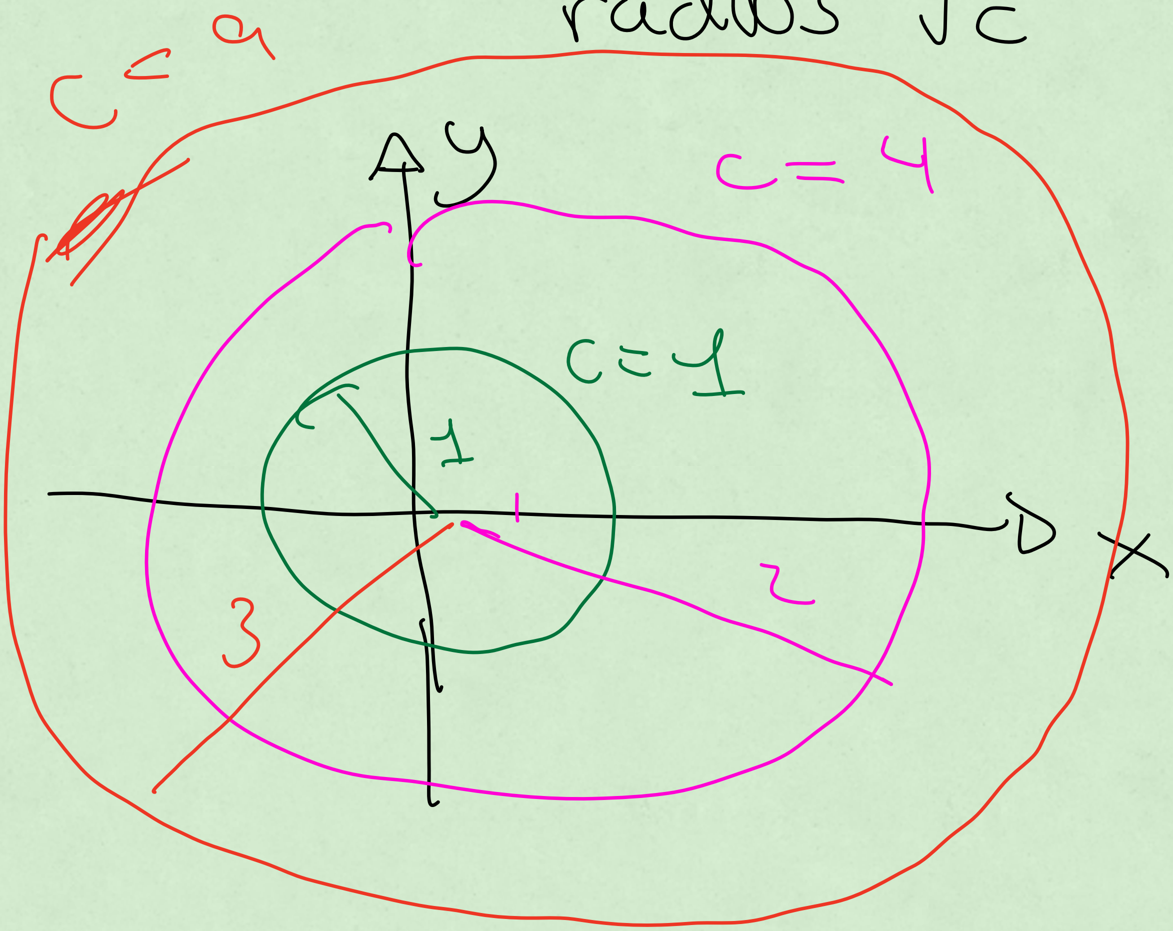
example:
level curves of

$$f(x,y) = x^2 + y^2$$



you solve

$$\underbrace{x^2+y^2 = c}_{\text{circle of radius } \sqrt{c}}$$



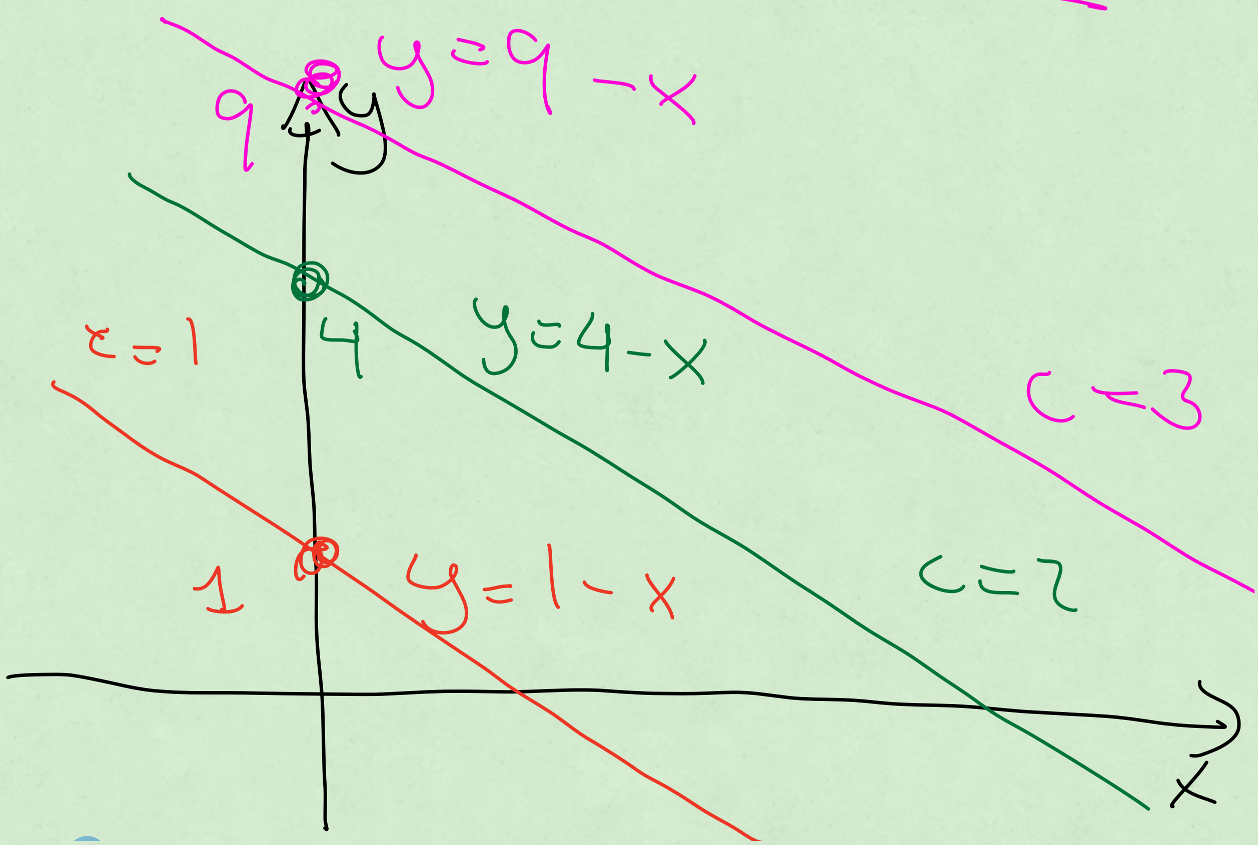
level curves

$$f(x,y) = \sqrt{x+y}$$

$$\sqrt{x+y} = c$$

$$x+y = c^2$$

$$\boxed{y = c^2 - x}$$



[redacted]

Level surface for

$$\underbrace{f(x,y,z)}_{\text{3 variables}}$$

we can't see the graph because we would need a fourth dimension

Level surface

$$\boxed{f(x,y,z) = c}$$

all the points (x, y, z)
where "f" has value "c"
↳ looks like
surfaces.

Lecture 10 (14.3 – 14.4)

Last time

– Level curves of $f(x,y)$

Points (x,y) where $f(x,y)$ has the same value "c"

$f(x,y) = x^2+y^2$

level curves

$x^2+y^2 = c$

↳ circles of radius $\sqrt{c}$

on each circle the value of "f" is the same

– Level surface of $f(x,y,z)$

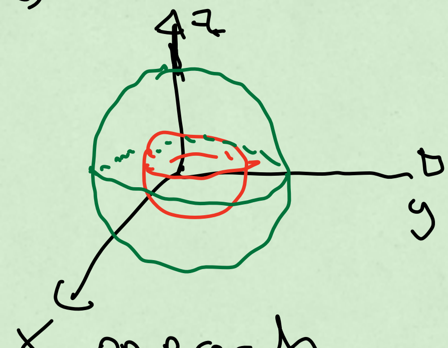
points (x,y,z) where $f(x,y,z)$ has the same value "c"

$f(x,y,z) = x^2+y^2+z^2$

level surface

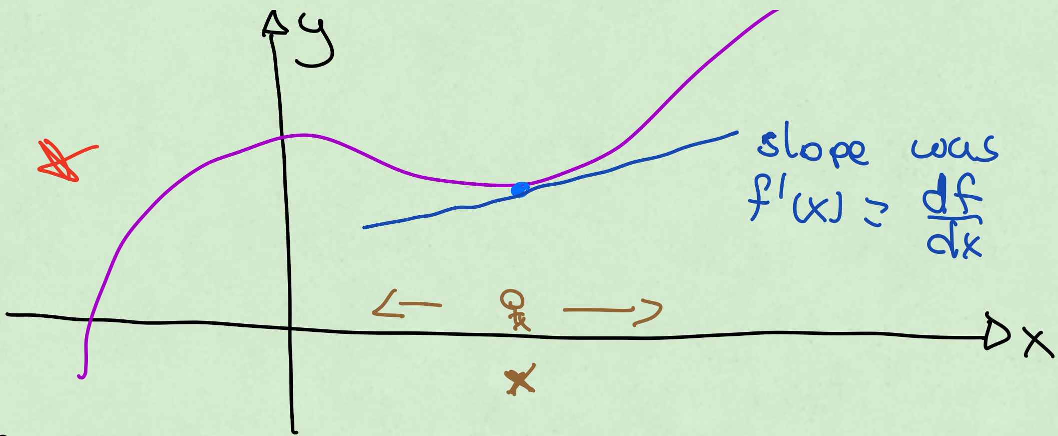
$x^2+y^2+z^2 = c$

↳ sphere radius $\sqrt{c}$



on each sphere the value of f remains constant

Partial Derivatives



$\frac{df}{dx}$ = notation reminds you of a fraction or ratio

↳ instantaneous rate of change

$$\boxed{\text{units of } \frac{df}{dx}} = \frac{\text{units of "f"}}{\text{units of "x"}}$$

$$\boxed{\text{units of } \frac{dx}{dt}} = \frac{\text{units of } x}{\text{units of } t}$$

$$\text{units of } \frac{d^2f}{dx^2} = \frac{\text{units of } f}{(\text{units of } x)^2}$$

$\frac{dx}{dt}$

Partial derivatives

$$f(x,y) = x^2 + y$$

$\frac{\partial f}{\partial x}$ = partial derivative of f with respect to x

= pretend "y" is constant when taking the derivative

$= 2x + 0$

$= 2x$

$\frac{\partial f}{\partial y}$ = partial derivative with respect to y

= pretend "x" is constant when taking derivatives

$= 0 + 1 = 1$

example:

$$f(x,y) = x^2 y^3$$

$$\frac{\partial f}{\partial x} = 2x y^3 = 2y^3 x$$

$$\frac{\partial f}{\partial y} = 3x^2 y^2$$

$$f(x,y) = \sin(x^2+xy)$$

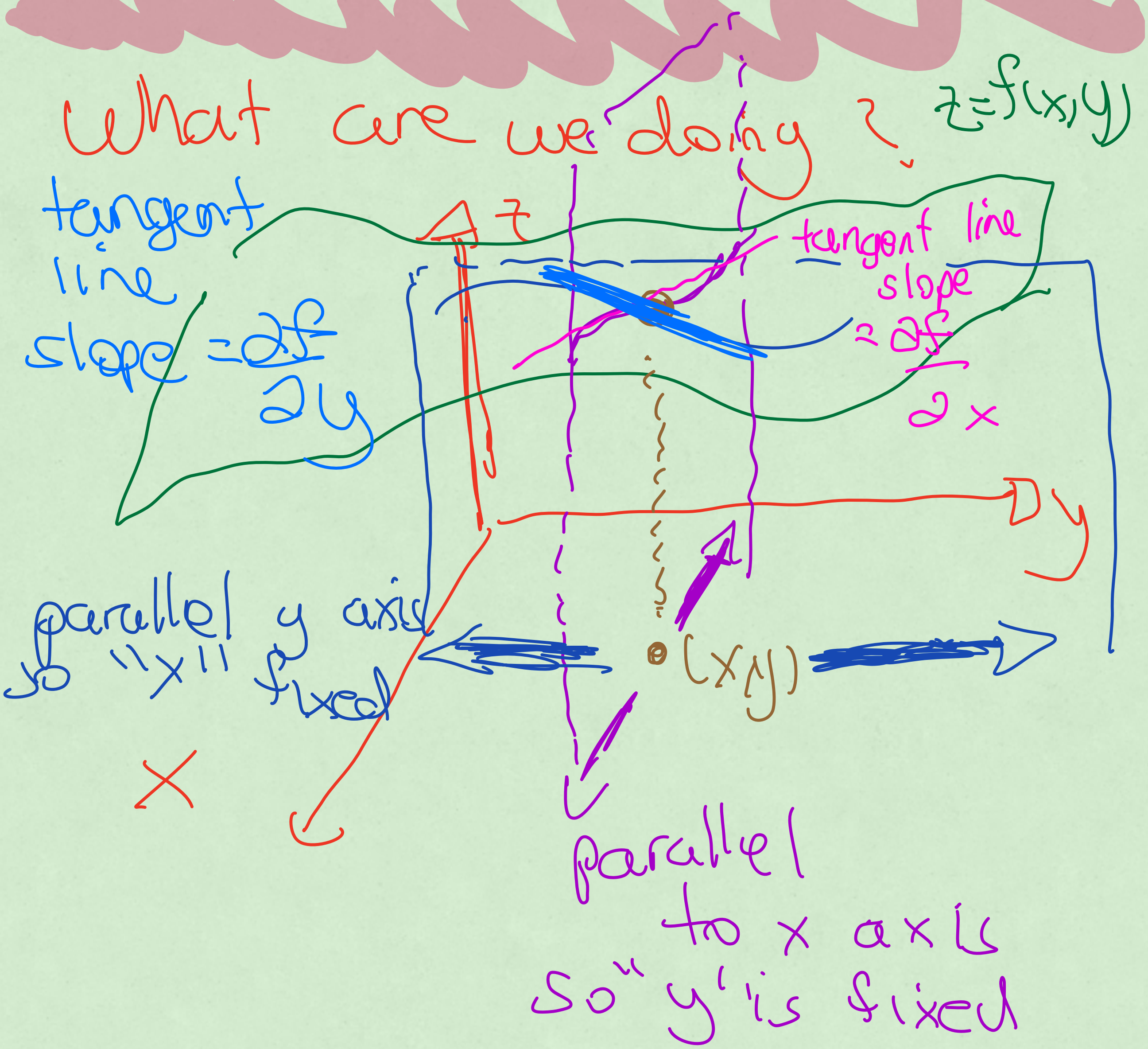
$$\frac{\partial f}{\partial x} = \cos(x^2+xy) \cdot \frac{\partial}{\partial x}(x^2+xy)$$

$$= (2x+y)\cos(x^2+xy)$$

$$\frac{\partial f}{\partial y} = \cos(x^2+xy)\cdot\frac{\partial}{\partial y}(x^2+xy)$$

$$= x\cos(x^2+xy)$$

What are we doing?



$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are slopes of tangent lines to curves obtained by intersecting the graph of the function with planes parallel to the xz and yz planes respectively

units of $\frac{\partial f}{\partial x} = \frac{\text{units of } f}{\text{units of } x}$

units of $\frac{\partial f}{\partial y} = \frac{\text{units of } f}{\text{units of } y}$

Second Partial Derivatives

$$f(x,y) = x^2 y^3$$

$$\frac{\partial f}{\partial x} = 2xy^3 \qquad \frac{\partial f}{\partial y} = 3x^2y^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy^3\right) = 2y^3$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy^3\right) = 6xy^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x^2y^2\right) = 6xy^2$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(3x^2y^2\right) = 6x^2y$$

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

In general,
the mixed partial derivatives agree, so when finding partial derivatives with respect to different variables, you can find the derivatives

in any order that you want.

Chain Rule

$$f(x) = e^x \qquad x = t^2 \qquad f(t) = e^{t^2}$$

$$\frac{df}{dt} = \frac{df}{dx}\,\frac{dx}{dt} = e^x\, 2t = 2te^{t^2}$$

$$\begin{tikzcd}
f \arrow[d, "\frac{df}{dx}"] \arrow[dd, bend right, "\frac{df}{dt}"'] \\
x \arrow[d, "\frac{dx}{dt}"] \\
t
\end{tikzcd}$$

$$\boxed{\frac{df}{dt} = \frac{df}{dx}\,\frac{dx}{dt}}$$

Chain Rule of twovariable

$$f(x,y) = xy \qquad x = t + s \qquad y = t^2 s$$

$$f(t,s) = (t+s)(t^2 s)$$

$$\boxed{f(t,s) = t^3 s + t^2 s^2}$$

$$\frac{\partial f}{\partial t} = 3t^2 s + 2ts^2$$

$$\frac{\partial f}{\partial s} = t^3 + 2t^2 s$$

$$
\begin{tikzcd}
 & & f \arrow[dl, "\frac{\partial f}{\partial x}"'] \arrow[dr, "\frac{\partial f}{\partial y}"] & & \\
 & x \arrow[dl, "\frac{\partial x}{\partial t}"'] \arrow[d, no head] & & y \arrow[d, no head] \arrow[dr, "\frac{\partial y}{\partial t}"] & \\
t & s & & s & t
\end{tikzcd}
$$

$$f = xy, \quad x = t+s, \quad y = t^2 s$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$= y \cdot 1 + x \cdot 2ts$$

$$= y + 2ts\, x$$

$$= t^2 s + 2ts(t+s)$$

$$= t^2 s + 2t^2 s + 2ts^2$$

$$\frac{\partial f}{\partial t} = 3t^2 s + 2ts^2$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Lecture 11 (14.5 – 14.6)

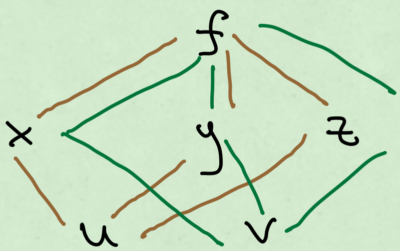
Directional derivative and gradient

Last time : chain rule

$$f(x,y,z) = x + y^2 + xz$$

$$x = u^2, \quad y = u+v, \quad z = uv$$

Find $\boxed{\dfrac{\partial f}{\partial u}}$ and $\boxed{\dfrac{\partial f}{\partial v}}$.



$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$= (1+z)2u + 2y \cdot 1 + x\,v$$

$$\boxed{\frac{\partial f}{\partial u} = (1+uv)2u + 2(u+v) + u^2 v}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$= (1+z)\cdot 0 + 2y \cdot 1 + x\,u$$

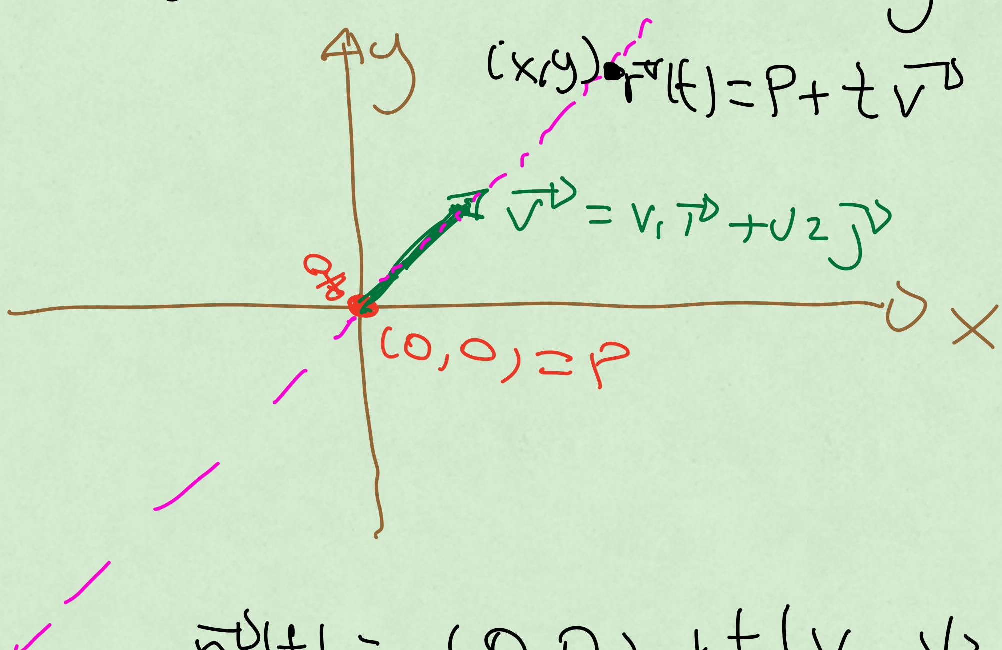
$$\boxed{\frac{\partial f}{\partial v} = 2(u+v) + u^3}$$

Directional Derivative

$f(x,y)$ = temperature at the point (x,y)

$\frac{\partial f}{\partial x}$ = rate of change of temperature if you move parallel to "x" axis

$\frac{\partial f}{\partial y}$ = rate of change of temperature if you move parallel to "y" axis.



$$\vec{r}(t) = (0,0) + t(v_1, v_2)$$
$$(x,y) = (tv_1, tv_2)$$

$$\begin{cases} x = tv_1 \\ y = tv_2 \end{cases}$$

On this line, f can be thought as a function of t.

$$\begin{tikzcd} & f \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & t & \end{tikzcd}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} v_1 + \frac{\partial f}{\partial y} v_2$$

dt

$$\frac{df}{dt} = \underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}_{\text{called gradient of "f"}} \cdot \underbrace{(v_1, v_2)}_{\vec{v}}$$

Gradient of f(x,y)

$$\text{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

examples

$f(x,y) = x^2 + xy$

gradient?

$$\frac{\partial f}{\partial x} = 2x+y$$

$$\frac{\partial f}{\partial y} = x$$

$$\nabla f = \operatorname{grad} f = (2x+y, x)$$

$$g(x,y) = x+\sin(y)$$

$$\frac{\partial g}{\partial x} = 1 \quad , \quad \frac{\partial g}{\partial y} = \cos(y)$$

$$\nabla g = \operatorname{grad}(g) = (1, \cos(y))$$

Gradient of $f(x,y,z)$

$$\nabla f = \text{grad } f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

example

$$f(x,y,z) = x + y^2 + xz$$

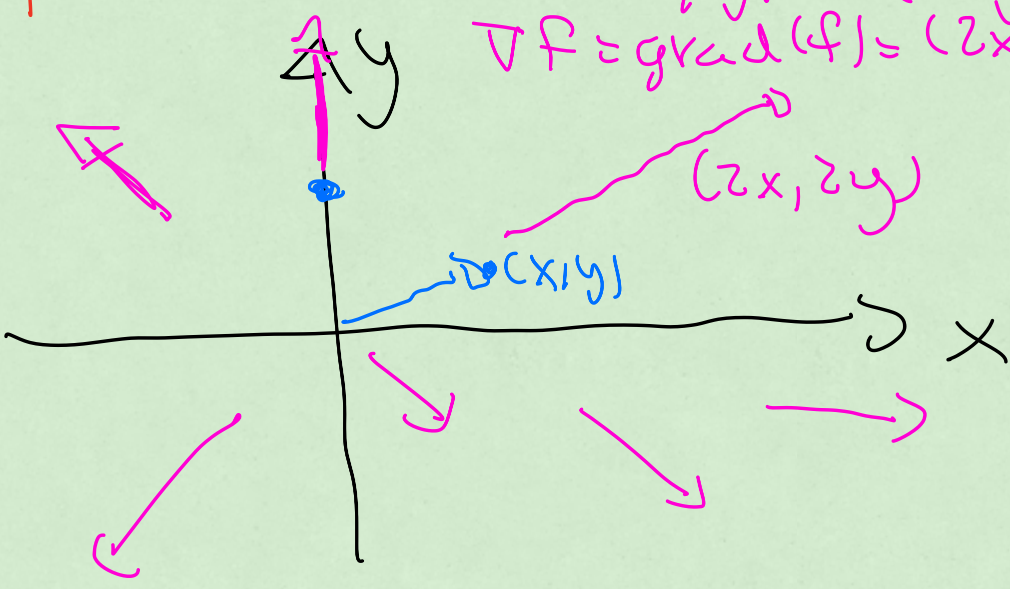
$$\frac{\partial f}{\partial x} = 1 + z, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = x$$

$$\nabla f = \text{grad} f = (1+z, 2y, x)$$

Gradient is a vector field, meaning, at the point (x,y) [or (x,y,z)] you draw a vector whose entries are the entries of grad f evaluated at the point in question

$f(x,y) = x^2+y^2$

$\nabla f = \text{grad}(f) = (2x,2y)$



Directional derivative of a function "f" in the direction of a vector $\vec{v}$

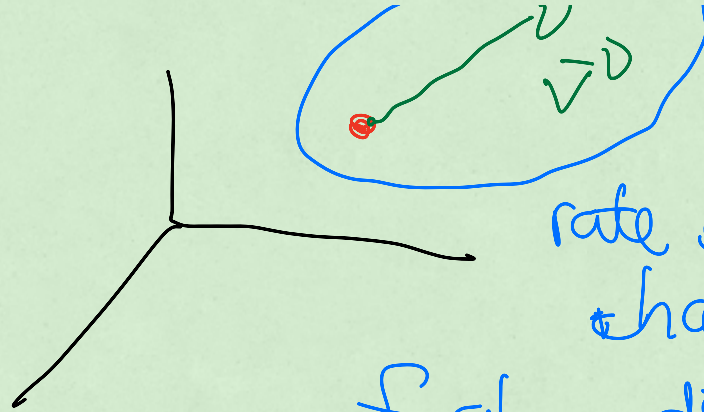
= rate of change of "f" in the direction determined by the vector $\vec{v}$.

$D_{\vec{v}} f$ = symbol for the directional derivative of f in the direction $\vec{v}$

conceptual question

$$D_{\vec{i}} f = \frac{\partial f}{\partial x} \qquad D_{\vec{k}} f = \frac{\partial f}{\partial z}$$

$$D_{\vec{j}} f = \frac{\partial f}{\partial y}$$



rate of change of f along direction $\vec{v}$

key formula

$$D_{\vec{v}} f = \nabla f \cdot \vec{v}$$

when you use this formula you have to use a vector $\vec{v}$ of norm 1. If you are given a vector which is not of norm 1, then really the formula is

$$D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|}$$

example :

Find the directional derivative of $f(x,y) = x^2 + xy$ in the direction of the vector $\vec{v} = (1, \sqrt{3})$ evaluated at the point $P = (2, 2)$

$$\nabla f = (2x + y, x)$$

$$D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$D_{\vec{v}} f = (2x+y, x) \cdot \frac{(1, \sqrt{3})}{\sqrt{1^2+3}}$$

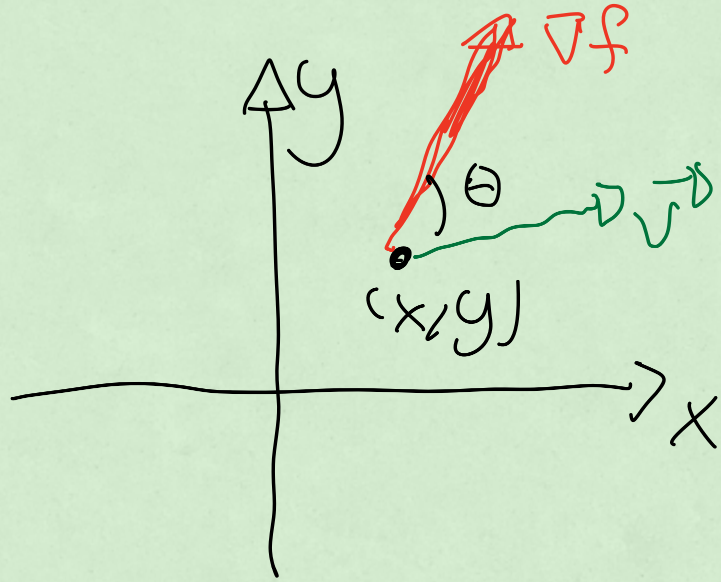
$$D_{\vec{v}} f = \underbrace{\frac{2x+y+\sqrt{3}x}{2}}$$

rate of change of "f" at the point (x,y) in the direction of $\vec{v} = (1, \sqrt{3})$

$$D_{\vec{v}} f(2,2) = \frac{4+2+2\sqrt{3}}{2}$$

$$= 3 + \sqrt{3}$$

Interpretation of directional derivative



$$D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$D_{\vec{v}} f = |\nabla f| \, \underbrace{\boxed{\left|\frac{\vec{v}}{|\vec{v}|}\right|}}_{1} \cos\theta$$

$$\boxed{D_{\vec{v}} f = |\nabla f| \cos\theta}$$

θ = angle between $\vec{v}$ and ∇f

$$\underbrace{-|\nabla f|}_{\text{when } \cos\theta = -1} \le D_{\vec{v}} f \le \underbrace{|\nabla f|}_{\text{when } \cos\theta = 1}$$

when $\vec{v}$ and ∇f point in opposite directions $\theta = \pi$

when $\vec{v}$ and ∇f are parallel $\theta = 0$

$\vec{v}$

∇f

$\vec{v}$

∇f

$$D_{\vec{v}} f \overset{\text{largest increase of } f}{=} |\nabla f| \Rightarrow \text{direction of } \nabla f$$

∇f

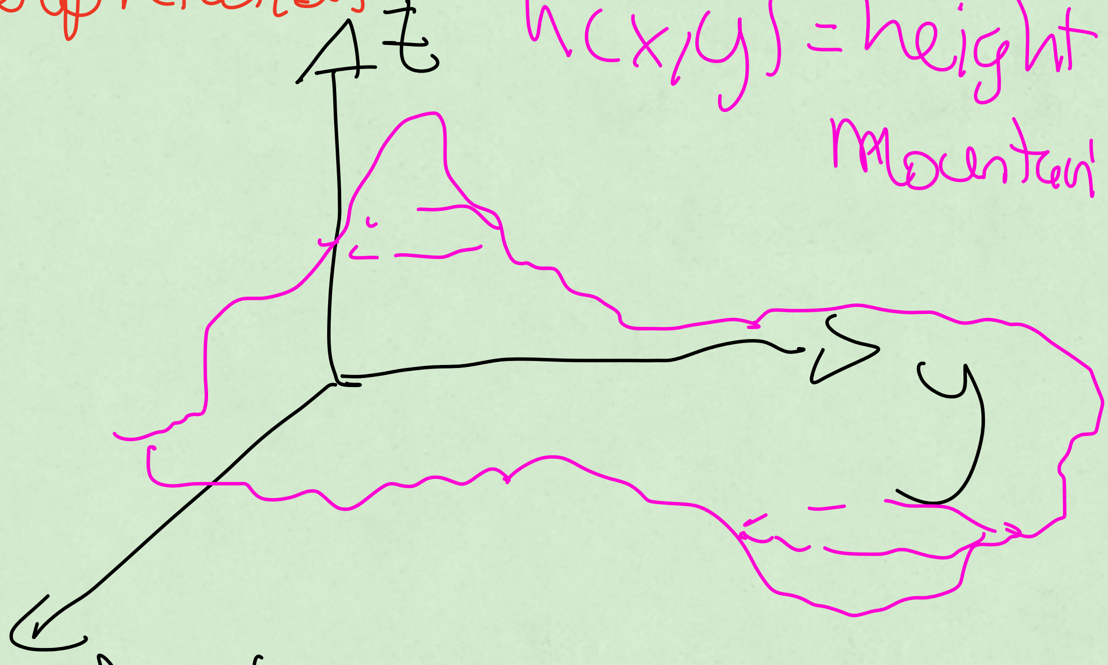
↳ largest decrease of $f = -|\nabla f|$

$\Rightarrow$ direction of $-\nabla f$.

interpretation

$h(x,y)$ = height

mountain



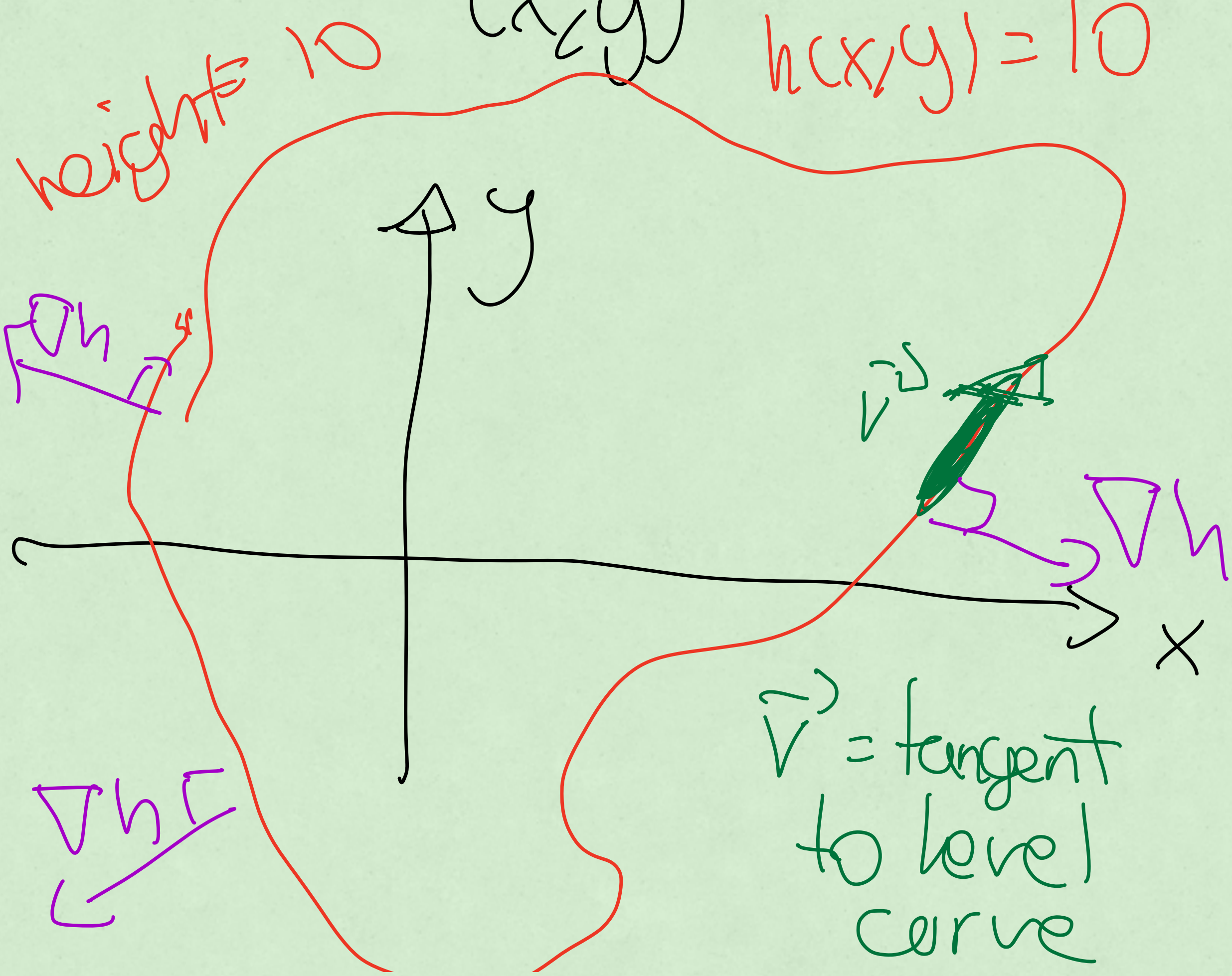
direction given by ∇h = steepest ascent

direction given by $-\nabla h$

= steepest descent

Back to level curves

$h(x,y)$ = height of mountain at (x,y)



$$D_{\vec{v}} h = 0 = \text{height}$$

is constant on level curve

$$\Downarrow$$

$$\nabla h \cdot \frac{\vec{v}}{|\vec{v}|} = 0$$

$$\nabla h \cdot \vec{v} = 0$$

$\rightarrow$ gradient ∇h is a vector perpendicular to the level curve

Lecture 12 (14.6-14.7)

Random topic (implicit differentiation)

calc 1:

$x^2+y^2=1 \longrightarrow y(x) = \pm\sqrt{1-x^2}$

find $\frac{dy}{dx}$ directly

$\downarrow$

indirect route

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

calc 3 version

$x^2+y^2+z^2 = 1 \longrightarrow z(x,y) = \pm\sqrt{1-x^2-y^2}$

find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ directly

indirect approach $\downarrow$ Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$\frac{\partial}{\partial x}(x^2+y^2+z^2) = \frac{\partial}{\partial x}(1)$$

$$2x + 0 + 2z\frac{\partial z}{\partial x} = 0$$

$$x + z\frac{\partial z}{\partial x} = 0$$

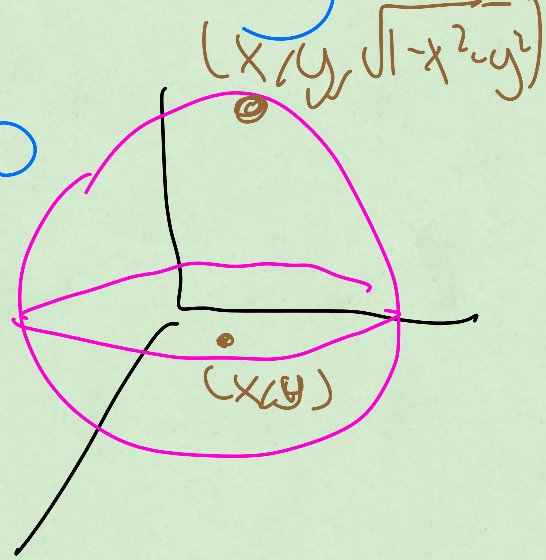
$\frac{\partial y}{\partial x} = 0$ because x, y

$$\boxed{\frac{\partial z}{\partial x} = -\frac{x}{z}}$$

are independent variables

$$\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right) = \frac{\partial}{\partial y}(1)$$

$$0 + 2y + 2z\frac{\partial z}{\partial y} = 0$$



$$\boxed{\frac{\partial z}{\partial y} = -\frac{y}{z}}$$

alternatively, can think of "y" as a function of x, z.

$$\left[\, y(x,z) = \pm\sqrt{1-x^2-z^2} \,\right]$$

$$x^2 + y^2 + z^2 = 1$$

$$\frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) = \frac{\partial}{\partial x}(1)$$

$$2x + 2y\frac{\partial y}{\partial x} + 0 = 0$$

$$\boxed{\frac{\partial y}{\partial x} = -\frac{x}{y}}$$

because x, z are now ind. variables

$$x^2 + y^2 + z^2 = 1$$

$$\frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right) = \frac{\partial}{\partial z}(1)$$

$$0 + 2y\frac{\partial y}{\partial z} + 2z = 0$$

$$\boxed{\frac{\partial y}{\partial z} = -\frac{z}{y}}$$

last time

$$D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$D_{\vec{v}} f = \text{grad } f \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$\nabla f = \operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

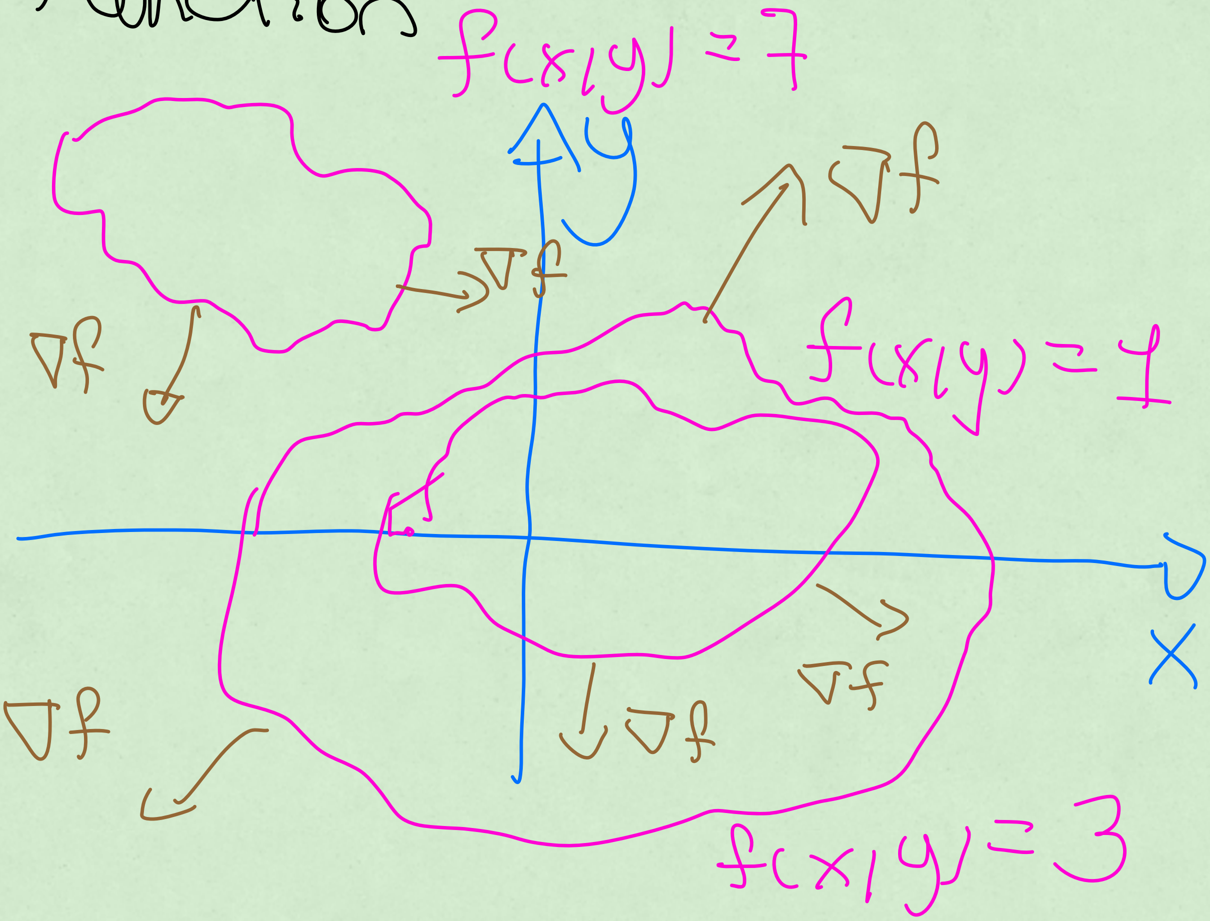
for $f(x,y,z)$

or $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$

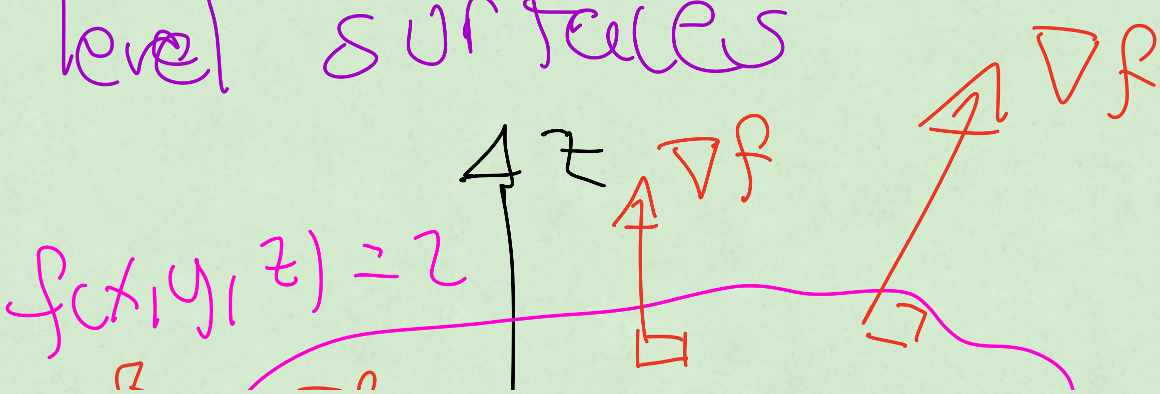
for $f(x,y)$

Important thing from last time!

Gradient ∇f is a vector which is perpendicular to the level curves of a function

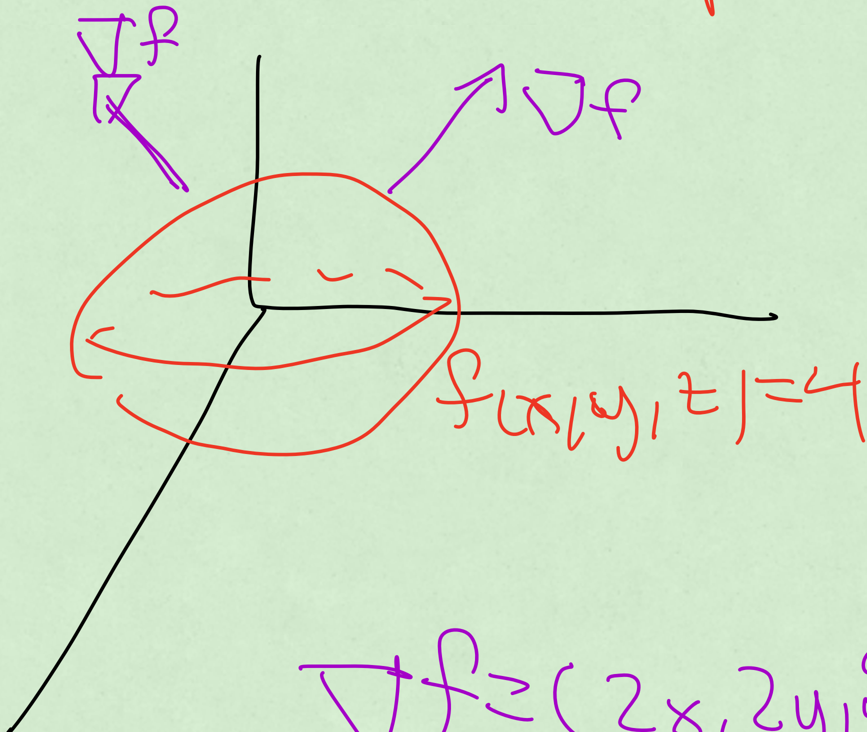


Same happens for level surfaces



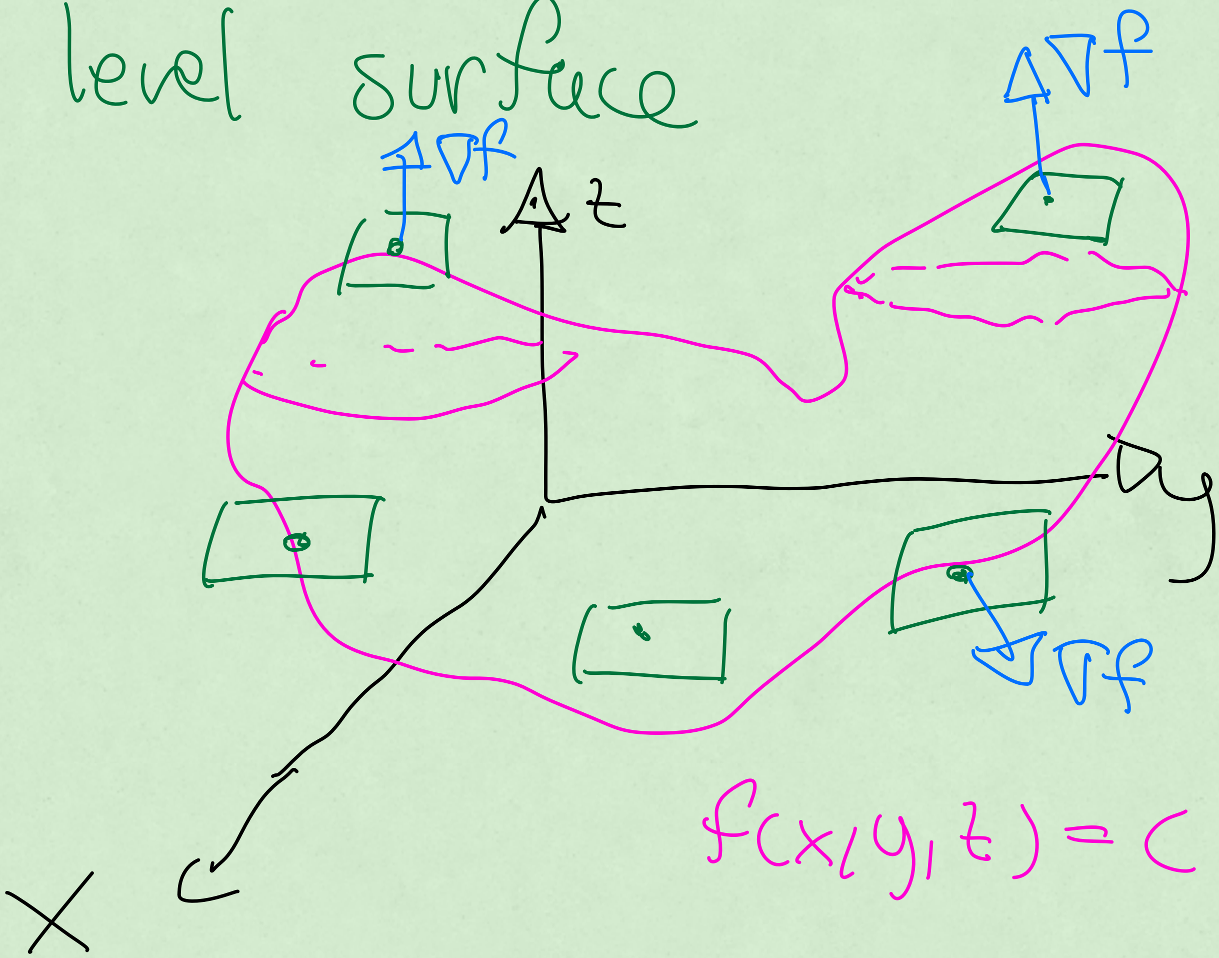
$$f(x,y,z) = x^2 + y^2 + z^2$$

levelsurface are spheres



$$\nabla f = (2x, 2y, 2z)$$

Tangent planes to a level surface



key idea: to find the equation of a tangent plane, you need a normal vector, and for this you can take the gradient

$$\vec{n} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

example:

$f(x,y,z) = xy - 2z^2$

level surface $f = 0$

$$xy - 2z^2 = 0$$

find tangent plane to this surface at the point

$P = (1, 2, 1)$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\nabla f = (y, x, -4z)$

$\nabla f(1,2,1) = \underbrace{(2, 1, -4)}_{\vec{n}}$

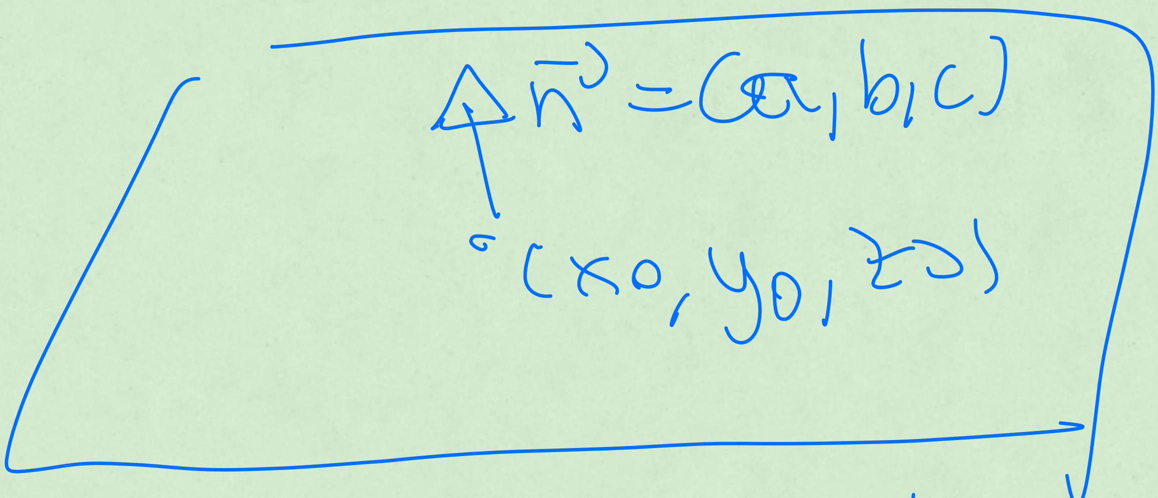
$2x + y - 4z = \vec{n} \cdot P$

$2x + y - 4z = (2,1,-4) \cdot (1,2,1)$

$$2x+y-4z = 2+2-4$$

$$2x+y-4z=0$$

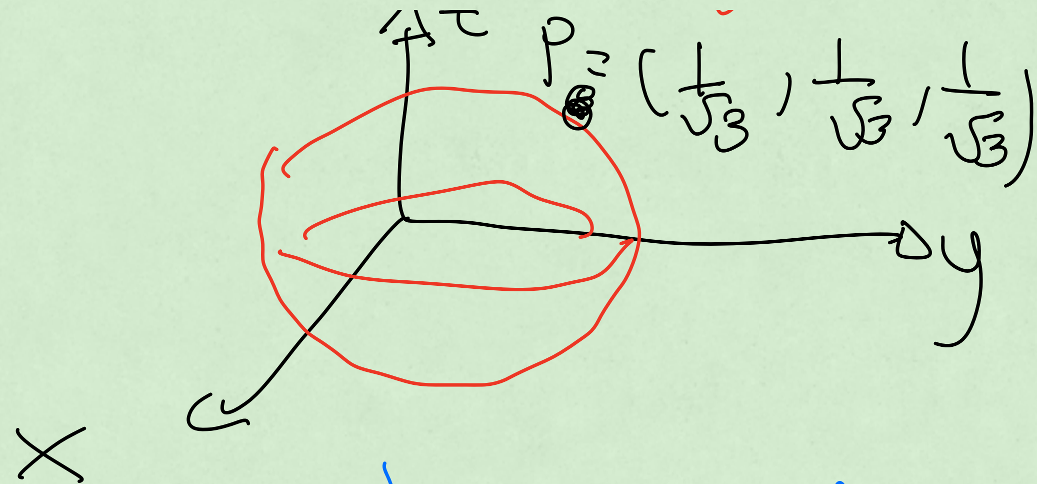
eq tangent plane



$$ax+by+cz = ax_0+by_0+cz_0$$

$$(a,b,c)\cdot(x,y,z) = (a,b,c)\cdot(x_0,y_0,z_0)$$

another example $x^2+y^3+z^2=1$



tangent plane at P

$$f(x,y,z) = x^2+y^2+z^2$$

$$\nabla f = (2x, 2y, 2z)$$

$$\nabla f(P) = \underbrace{\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)}_{\vec{n}}$$

$$\frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y + \frac{2}{\sqrt{3}}z$$

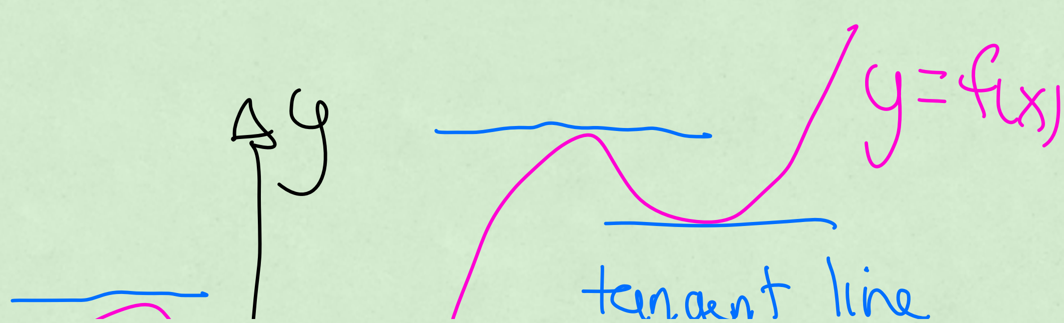
$$(1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3}) \cdot \left(2/\sqrt{3}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$$

$$= \frac{2}{3} + \frac{2}{3} + \frac{2}{3}$$

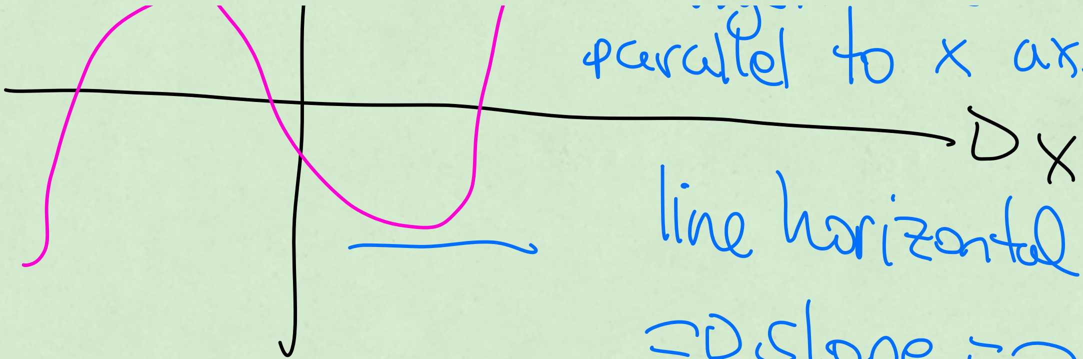
$$\frac{2x}{\sqrt{3}} + \frac{2y}{\sqrt{3}} + \frac{2z}{\sqrt{3}} = 2$$

$$\boxed{x + y + z = \sqrt{3}}$$

14.7 Optimization



parallel to x axis

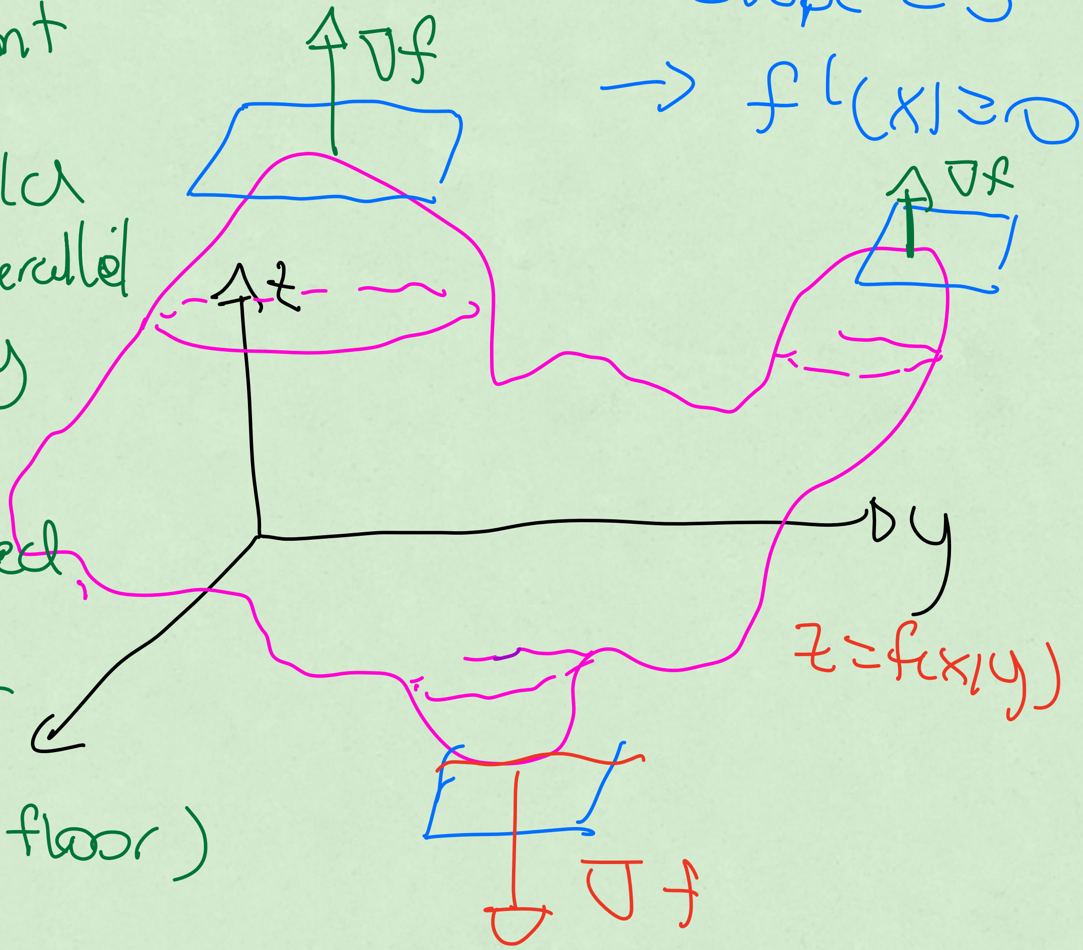


line horizontal

$\Rightarrow$ slope $= 0$

$\rightarrow f'(x) = 0$

tangent plane should be parallel to xy plane (not tilted with respect to the floor)



So ∇f should be parallel to the vector $\vec{k} = (0,0,1)$

So we need

$$\boxed{\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0}$$

Critical point of $z = f(x,y)$

It is a point where

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

---

example ;

$f(x,y) = xy$

Critical points

$$\frac{\partial f}{\partial x} = 0 \rightarrow y = 0$$

$$\begin{cases} \dots \\ \dfrac{\partial f}{\partial y} = 0 \rightarrow x = 0 \end{cases}$$

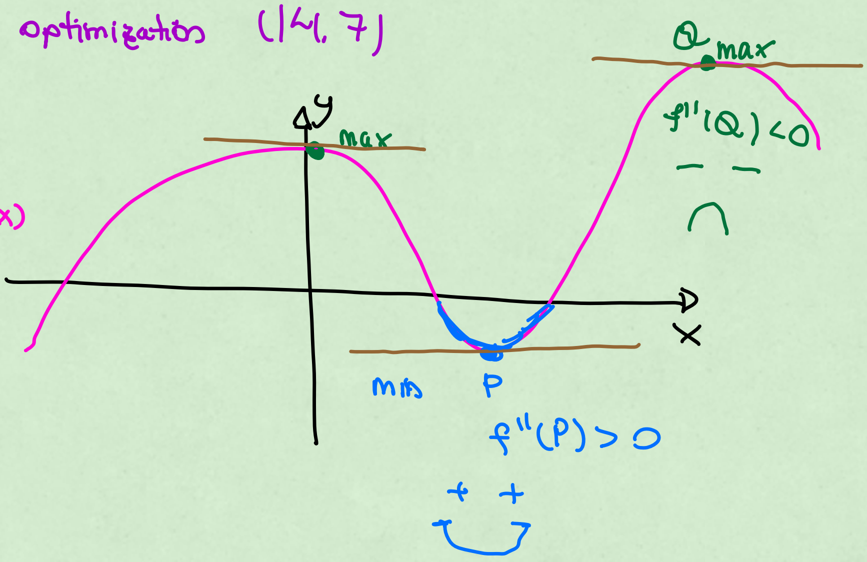
so $(0,0)$ is the only critical point for $f(x,y) = xy$

Lecture 13: optimization (14.7)

Calc 1

$y=f(x)$

critical points: $f'(x)=0$



Critical points of $f(x,y)$

$\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

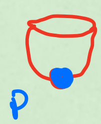
local max (top of a mountain)

as you approach this point your height increases

saddle (potato Pringles chip)

it behaves like a max or a min depending on how you approach it

local min (valley)



as you approach this point your height decreases

Second Derivative Test

① Find a critical point P: $\frac{\partial f}{\partial x}(P)=0$ and $\frac{\partial f}{\partial y}(P)=0$

② Compute the second partial derivatives and write them as a 2×2 matrix

$$H=\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

(H here stands for Hessian)

$\det H(P) = 0$ [game over]
this test is inconclusive and you gain no information

$\underbrace{\det H(P)}$
value of determinant of this matrix at the critical point (this is a number)

$\det H(P) < 0$ (negative)
you get a saddle point

$\det H(P) > 0$
positive det

$\frac{\partial^2 f}{\partial x^2}(P) > 0$
$++$ P is a relative min

$\frac{\partial^2 f}{\partial x^2}(P) < 0$
$--$ P is a relative max

Example:

$$f(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$$

critical points (finding)

$$\frac{\partial f}{\partial x} = x^2 + 2y - 6$$

$$\frac{\partial f}{\partial y} = 2y + 2x - 3$$

$$\begin{cases} x^2 + 2y - 6 = 0 & (1) \\ 2y + 2x - 3 = 0 & (2) \end{cases}$$

start with (2)

$$\boxed{2y = 3 - 2x}$$

$$\boxed{y = \frac{3}{2} - x}$$

substitute in (1)

$$x^2 + (3 - 2x) - 6 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$x = 3$

$y = \frac{3}{2} - 3 = -\frac{3}{2}$

point

$\left(3, -\frac{3}{2}\right)$

$x = -1$

$y = \frac{3}{2} + 1 = \frac{5}{2}$

$\left(-1, \frac{5}{2}\right)$

found two critical points!!!

classify them

$$\frac{\partial f}{\partial x} = x^2 + 2y - 6$$

$$\frac{\partial f}{\partial y} = 2y + 2x - 3$$

$$\frac{\partial^2 f}{\partial x^2} = 2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\boxed{H = \begin{pmatrix} 2x & 2 \\ 2 & 2 \end{pmatrix}}$$

$\det H = 4x - 4 = 4(x-1)$

critical point $P = \left(3, -\frac{3}{2}\right)$

$\det H(P) = 4(3-1) = 8 > 0$

$\frac{\partial^2 f}{\partial x^2}(P) = 2x = 2 \cdot 3 = 6 > 0$

$+ \quad +$

$\smile$

P is relative min!

critical point $Q = \left(-1, \frac{5}{2}\right)$

$\det H(Q) = 4(x-1) = 4(-1-1) = -8 < 0$

Q is a

Saddle point!

Relative min = local min

Relative max = local max

$$f(x,y) = x^3 + y^3$$

critical points

$$\frac{\partial f}{\partial x} = 3x^2 \qquad \frac{\partial f}{\partial y} = 3y^2$$

$$\begin{cases} 3x^2 = 0 \\ 3y^2 = 0 \end{cases} \longrightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix}$$

only one $\boxed{(0,0)}$

classify it

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial^2 f}{\partial y^2} = 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$H = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$$

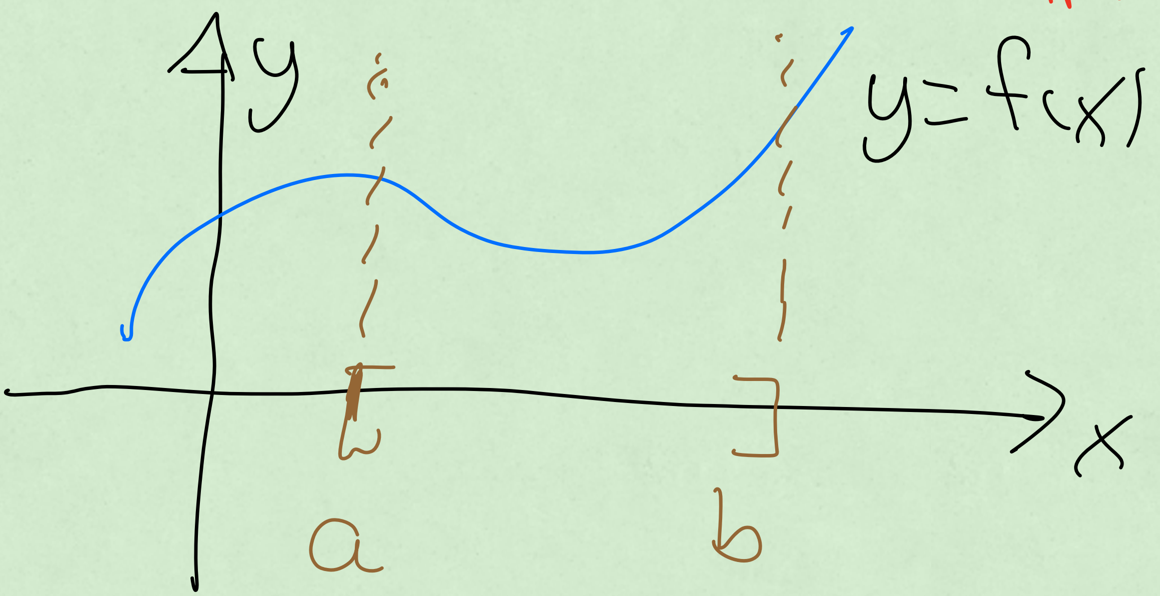
$$\det H = 36xy$$

$\det H(0,0) = 0$

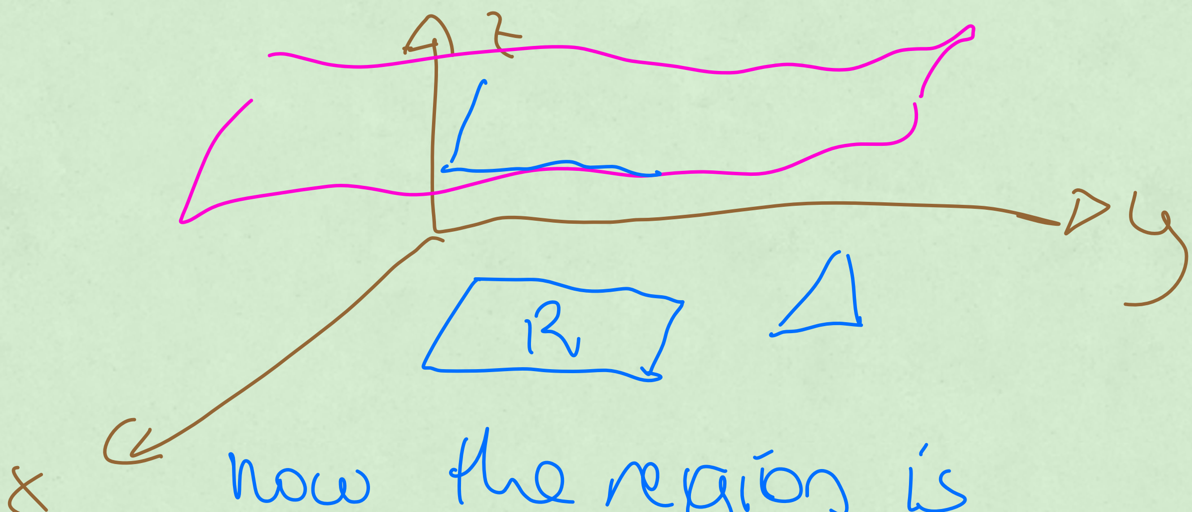
inconclusive, we can't classify it

now with a twist:

absolute max and min!



to find abs max and abs min here which includes the critical points of f(x) on this interval and the value of f(x) at the boundary points.



now the region is either a rectangle or a triangle.

- make a list which includes

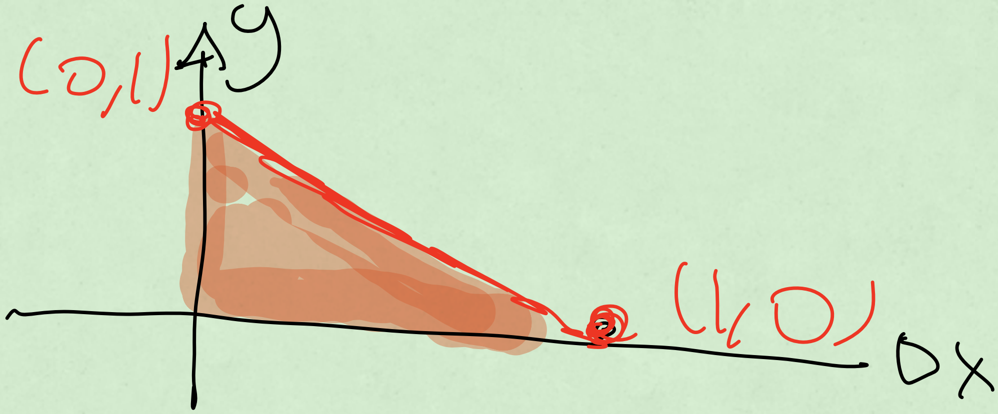
① the critical points inside the region

(2) the vertices of the rectangle or triangle

(3) The critical points that you get from each edge of the region

find abs max / abs min of

$$f(x,y) = 4x - 8xy + 2y + 1$$



critical points of f

$$\frac{\partial f}{\partial x} = 4 - 8y \qquad \frac{\partial f}{\partial y} = -8x + 2$$

$$\frac{\partial^2 f}{\partial x^2} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 0 \qquad \frac{\partial^2 f}{\partial x \partial y} = -8$$

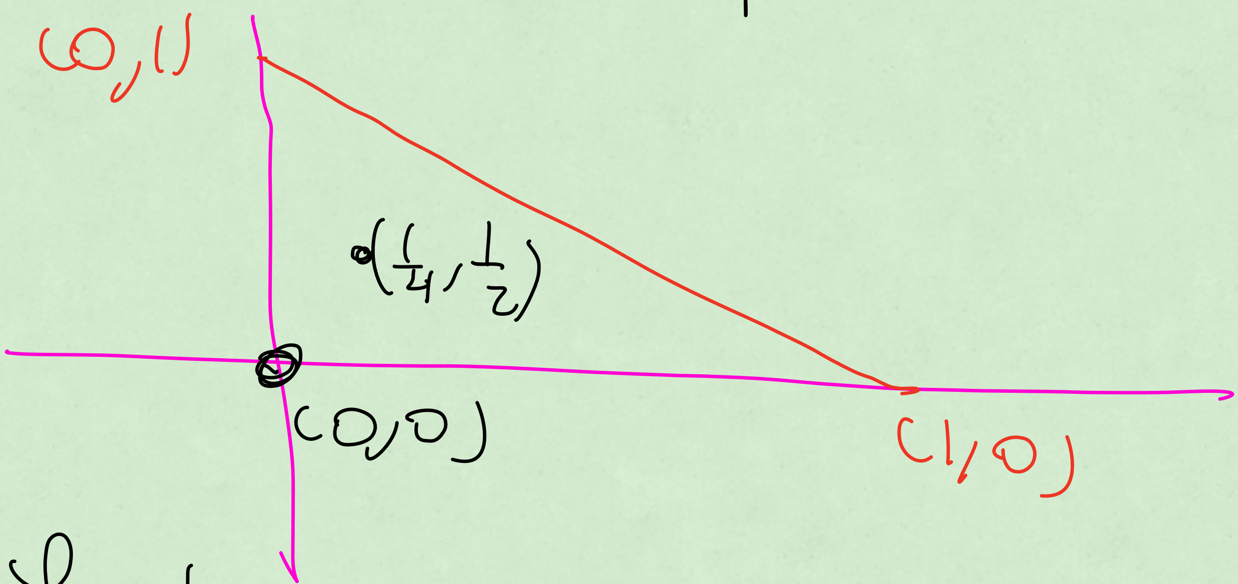
$$\begin{cases} 4 - 8y = 0 \to y = \frac{1}{2} \\ -8x + 2 = 0 \to x = \frac{1}{4} \end{cases}$$

$$\left(\frac{1}{4}, \frac{1}{2}\right) \to \text{saddle point}$$

$$H = \begin{pmatrix} 0 & -8 \\ -8 & 0 \end{pmatrix}$$

$$\det H\left(\frac{1}{4}, \frac{1}{2}\right) = -64$$

new part!



List:

① critical point $\left(\frac{1}{4},\frac{1}{2}\right)$

② vertices triangle:
(0,0), (1,0), (0,1)

③ critical points coming from each edge of the function

$$f(x,y) = 4x - 8xy + 2y + 1$$

Green side $x=0$

$f(y) = 2y+1$

$f'(y) = 2$ [no critical points]

Blue side $y=0$

$f(x) = 4x+1$

$f'(x) = 4$ [no critical points]

hypotenuse $y = 1 - x$

$f(x,y) = 4x - 8xy + 2y + 1$

$f(x) = 4x - 8x(1-x) + 2(1-x) + 1$

$= 4x - 8x + 8x^2 + 2 - 2x + 1$

$= 8x^2 - 6x + 3$

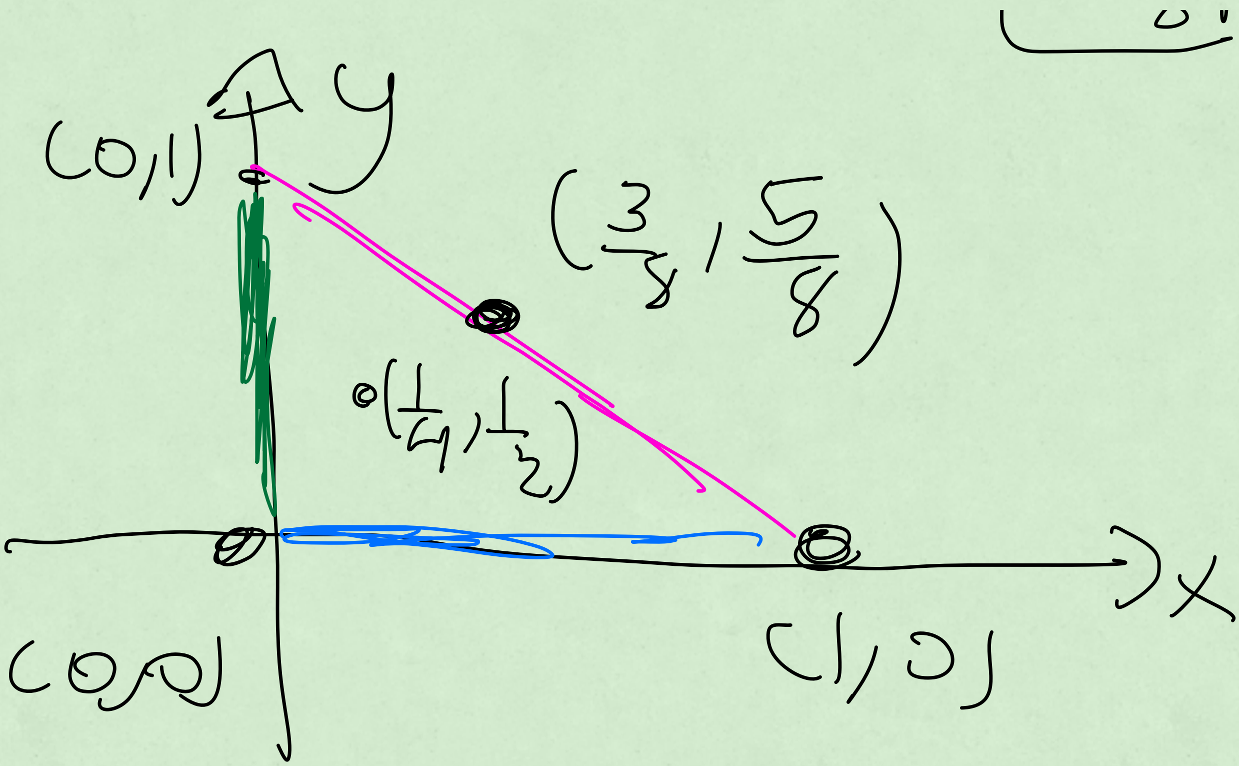
$f'(x) = 16x - 6 = 0$

when $16x = 6$

$\boxed{x = \frac{3}{8}}$ $y = 1 - x$

$= 1 - \frac{3}{8}$

$= \boxed{\frac{5}{8}}$



(x,y)	$f(x,y) = 4x - 8xy + 2y + 1$
$(\frac{1}{4}, \frac{1}{2})$	2
$(0,0)$	1
$(1,0)$	5
$(0,1)$	3

abs min

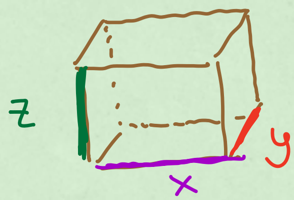
abs max

$$\left(\frac{3}{8}, \frac{5}{8}\right) \Big| \quad \frac{15}{8}$$

Lagrange Multipliers (14.8)

Problem: build a box with no top (open box) so that

- the surface of the box is $10 ft^2$.
- the volume of the box is a big as possible



Volume $V = xyz$

area of the box $= xy + 2xz + 2yz = 10$

goal: maximize volume V!

two ways to solve this problem!

Method 1: turn the volume into a function of two variables in order to apply the methods from before

(constraint equation)

$$xy + 2xz + 2yz = 10$$

$$2z(x+y) = 10 - xy$$

$$\boxed{z = \frac{10 - xy}{2(x+y)}}$$

$$V = xyz = \frac{xy(10-xy)}{2(x+y)} = \frac{10xy - x^2y^2}{2x+2y}$$

$$f(x,y) = \frac{10xy - x^2y^2}{2x+2y}$$

quotient rule

$$\frac{\partial f}{\partial x} = \frac{(10y - 2xy^2)(2x+2y) - 2(10xy - x^2y^2)}{(2x+2y)^2}$$

$$\frac{\partial f}{\partial x} = \frac{20xy + 20y^2 - 4x^2y^2 - 4xy^3 - 20xy + 2x^2y^2}{4(x+y)^2}$$

$$\frac{\partial f}{\partial x} = \frac{20y^2 - 2x^2y^2 - 4xy^3}{4(x+y)^2}$$

$$\frac{\partial f}{\partial x} = \frac{y^2(20 - 2x^2 - 4xy)}{4(x+y)^2}$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{y^2(10 - x^2 - 2xy)}{2(x+y)^2}}$$

wolfram alpha [:)]

$$\frac{\partial f}{\partial y} = \frac{x^2(10 - y^2 - 2xy)}{2(x+y)^2}$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$$

$$\begin{cases} \dfrac{y^2(10-x^2-2xy)}{2(x+y)^2} = 0 \\ \dfrac{x^2(10-y^2-2xy)}{2(x+y)^2} = 0 \end{cases}$$

numerators must be 0

$$\begin{cases} (1) & y^2(10-x^2-2xy) = 0 \\ (2) & x^2(10-y^2-2xy) = 0 \end{cases}$$

eq (1)

$$y^2(10-x^2-2xy) = 0$$

$y = 0$

volume of box $= 0$

not interesting for us!

or

$$10-x^2-2xy = 0$$

$$\boxed{10-x^2 = 2xy}$$

eq (2)

$$x^2(10-y^2-2xy) = 0$$

or

$x=0$

volume of box $=0$

not interesting for us!

$10-y^2-2xy=0$

$$\boxed{10-y^2=2xy}$$

$$10-x^2=10-y^2$$

$$0=x^2-y^2$$

$$0=(x-y)(x+y)$$

$$\boxed{x=y}$$

$$\boxed{x=-y}$$

not good since "x,y,z" are the lengths of the sites of a box

$$\boxed{x = y}$$

substitute in

$$10 - x^2 = 2xy$$

$$10 - x^2 = 2x^2$$

$$10 = 3x^2$$

$$\boxed{x = \frac{\sqrt{10}}{\sqrt{3}}}$$

$$\boxed{y = \frac{\sqrt{10}}{\sqrt{3}}}$$

$$z = \frac{10 - xy}{2(x+y)} = \frac{10 - \frac{10}{3}}{4\sqrt{10}}$$

$\sqrt{3}$

method 2: Lagrange multipliers

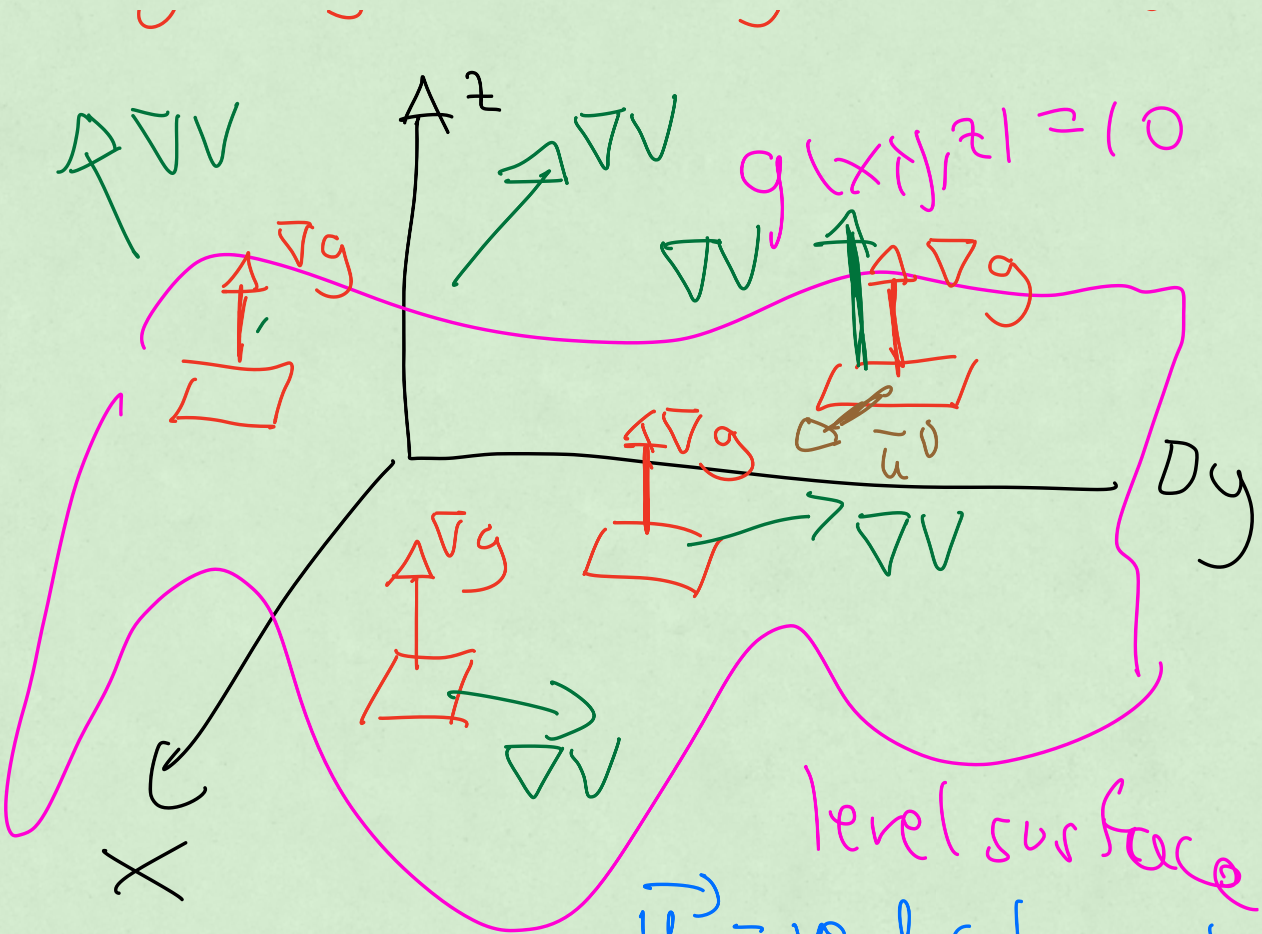
work with all the initial variables without trying to substitute one of them in terms of the others

$$V(x,y,z) = xyz$$

$$\boxed{xy + 2xz + 2yz = 10}$$

level surface of

$$g(x,y,z) = xy + 2xz + 2yz$$



maybe there is a place where ∇V and ∇g are parallel

$$\left[\ \nabla V = \underbrace{\lambda}_{\text{rescaling factor}} \nabla g\ \right]$$

a number

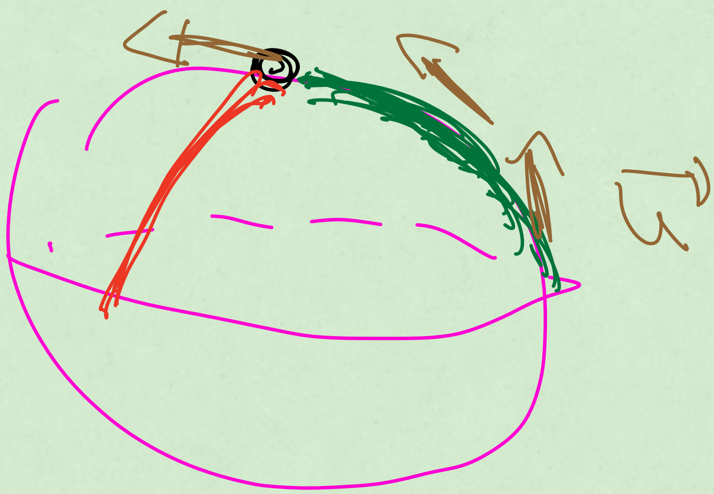
$$D_{\vec{u}} V = \nabla V \cdot \vec{u}$$
$$= \lambda \underbrace{[\nabla g \cdot \vec{u}]}_{0}$$

$$D_{\vec{u}} V = 0$$

so at this point the directional derivative vanishes so it is like a critical point.

$x^2+y^2+z^2=1$

$\vec{u}$ = the velocity vector of a path on the surface



if ∇V and ∇g are parallel at some point on the surface, then the directional derivative of $\vec{V}$ vanishes for all the paths on this surface at the point in question

method 2 :
you find when the two gradients are parallel

$$V(x,y,z) = xyz$$

$$g(x,y,z) = xy + 2xz + 2yz$$

$$\nabla V = (yz, xz, xy)$$

$$\nabla g = (y+2z, x+2z, 2x+2y)$$

when is ∇V parallel to ∇g?

constant number called Lagrange multiplier

$$\nabla V = \lambda \nabla g$$

$$(yz, xz, xy) = \lambda(y+2z, x+2z, 2x+2y)$$

$$\begin{cases} yz = \lambda(y+2z) & (1) \\ xz = \lambda(x+2z) & (2) \end{cases}$$

(3) $\left(xy = \lambda(2x + 2y) \right.$

(2) − (1):

$$xz - yz = \lambda x + \cancel{2\lambda z} - \lambda y - \cancel{2\lambda z}$$

$$z(x-y) = \lambda(x-y)$$

$$z(x-y) - \lambda(x-y) = 0$$

$$(x-y)(z-\lambda) = 0$$

$\boxed{x = y}$ or $\boxed{z = \lambda}$

you get to
the same
conclusion
as before

$x = \frac{\sqrt{10}}{\sqrt{3}}$, $y = \frac{\sqrt{10}}{\sqrt{3}}$

what happens if $\boxed{\lambda = z}$

go to first equation!

$$yz = \underbrace{\boxed{\lambda}}_{z}(y + 2z)$$

$$yz = zy + 2z^2$$

$$2z^2 = 0$$

$z = 0$ → Volume = 0

bad 1. (ignore this case)

Lagrange multiplier method

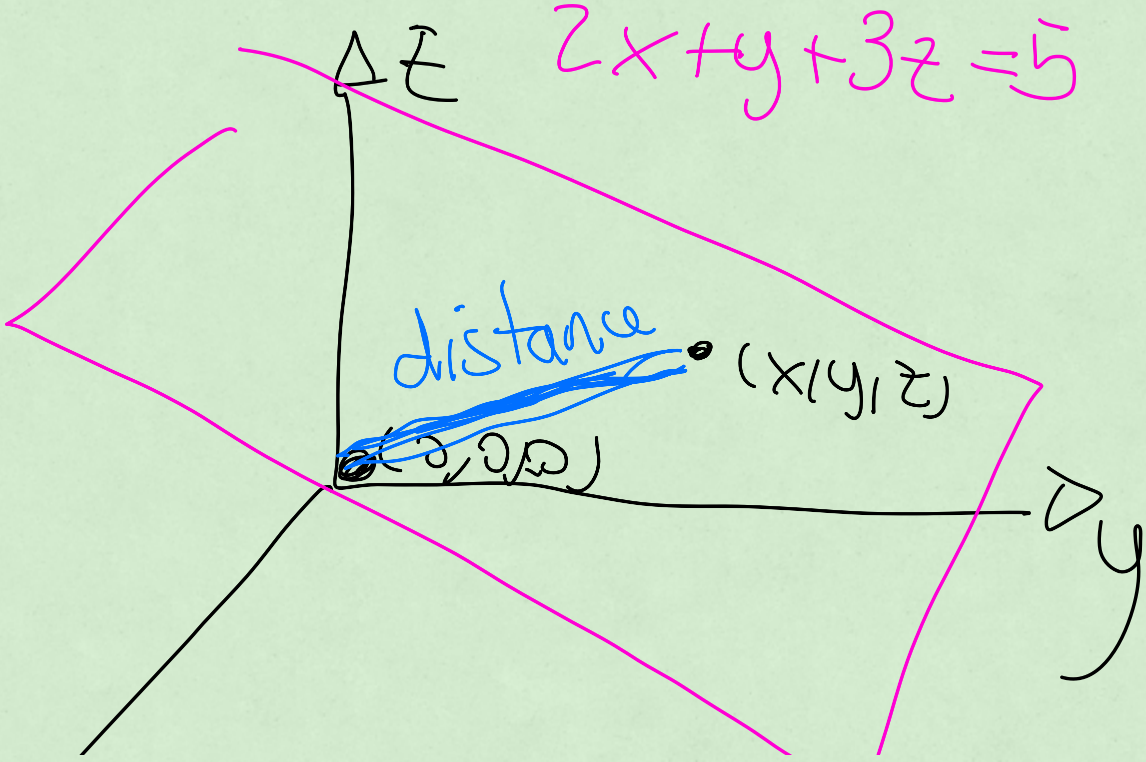
$f(x,y,z) \rightarrow$ you want to maximize or minimize this one (like the volume function)

$g(x,y,z) \rightarrow$ constraint equation (level surface) if you want to maximize or minimize f you must solve

$$\boxed{\nabla f = \lambda \nabla g}$$

λ = constant = Lagrange multiplier

$2x+y+3z=5$



x

$2x+y+3z=5$

Find point on the plane closest to the origin

distance $= \sqrt{x^2+y^2+z^2}$

$f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = 2x+y+3z=5$

Lagrange multiplier:

$$\nabla f = \lambda \nabla g$$

$$(2x, 2y, 2z) = \lambda(2, 1, 3)$$

$$\begin{cases} 2x = 2\lambda \rightarrow x = \lambda \\ 2y = \lambda \rightarrow y = \frac{\lambda}{2} \\ 2z = 3\lambda \rightarrow z = \frac{3\lambda}{2} \end{cases}$$

can use

$$2x + y + 3z = 5$$

$$2\lambda + \frac{\lambda}{2} + \frac{9\lambda}{2} = 5$$

$$\boxed{\lambda = \frac{5}{7}}$$

$x = \frac{5}{7}, \quad y = \frac{5}{14}, \quad z = \frac{15}{14}$

point

$$\left(\frac{5}{7}, \frac{5}{14}, \frac{15}{14}\right)$$

Remark: you don't need to check for these problems that the point or points you find yield a max or min without hagrange

$$z = \frac{5-2x-y}{3}$$

$$f(x,y,z) = x^2+y^2+z^2$$

$$f(x,y) = x^2+y^2+\left(\frac{5-2x-y}{3}\right)^2$$

look for

$$\frac{\partial f}{\partial x} = 0 \quad , \quad \frac{\partial f}{\partial y} = 0$$