Joint evolution of a Lorentz covariant massless scalar field and its point source on $\mathbb{R}^{1,1}$

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Introduction

We present a proof that the static solution of the Cauchy problem for a massless real scalar field that is sourced by a point charge in $1+1$ dimensions is asymptotically stable under perturbation by compactly-supported incoming radiation. This behavior is due to the process of back-reaction, the phenomenon in which a charged particle that is perturbed radiates energy until returning to its original state.

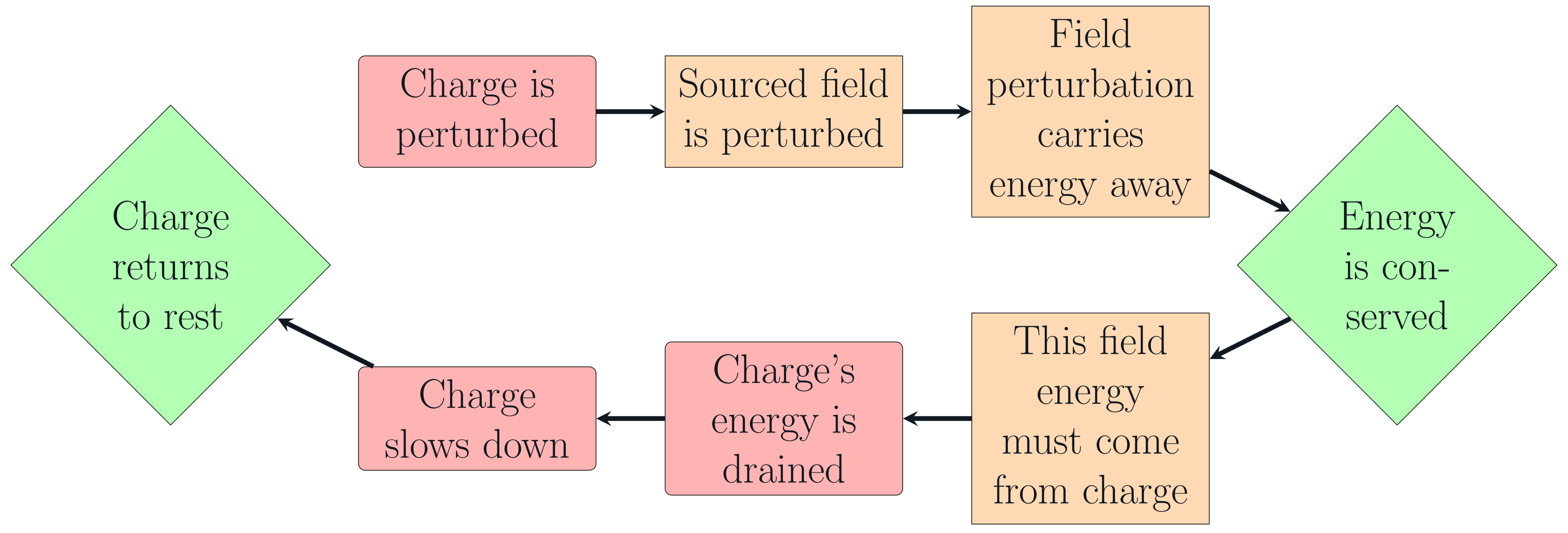

Fig. 1: Restorative effect of back-reaction.

Attempting to study the process of back-reaction in the framework of Maxwell Electrodynamics leads to an inconsistency, because the Lorentz force requires us to evaluate the electromagnetic field along the particle's path, and yet the field sourced by the particle is undefined precisely along this path. Resolving the radiation-reaction problem has been an open problem for more than a century.

Inconsistent Force Law From Action Principle

The action for a relativistic point charge coupled to a massless scalar field U on $\mathbb{R}^{1,1}$ is

$$S[U,z] = \int -(\tilde{m} - aU(z))\sqrt{\eta_{\mu\nu}\dot{z}^\mu\dot{z}^\nu}\,d\theta + \int \frac{1}{2}\eta^{\mu\nu}(\partial_\mu U)(\partial_\nu U)\sqrt{-\eta}\,dx^2 \tag{1}$$

Here z^μ, $\tilde{m}$, and a represent the particle's space-time position, bare mass, and scalar charge respectively, while θ is an arbitrary parameterization of the particle's worldline. Taking variations to extremize the action returns a contradiction:

$$\frac{\delta S}{\delta z^\mu} = 0 \quad\Downarrow\quad \frac{dp^\mu}{d\tau} = -a\partial^\mu U(z) \quad\Downarrow\quad \text{Need to evaluate } \partial^\mu U \text{ along } z$$

$$\frac{\delta S}{\delta U} = 0 \quad\Downarrow\quad \eta^{\mu\nu}\partial_\mu\partial_\nu U = a\int \delta^{(2)}(x - z(\tau))d\tau \quad\Downarrow\quad \text{Cannot evaluate } \partial^\mu U \text{ along } z$$

So there can be no joint evolution of the field-particle system which extremizes the action.

Force Law from Energy-Momentum Conservation

From the action, we can derive stress energy tensors for the particle and the scalar field, $T_p^{\mu\nu}$ and $T_S^{\mu\nu}$ respectively. If there were joint-evolutions which extremized the action, they would satisfy the law of energy-momentum conservation

$$\partial_\mu T_p^{\mu\nu} = -\partial_\mu T_S^{\mu\nu}. \tag{2}$$

We opt to derive our force law directly from energy-momentum conservation.

Theorem 1. *Suppose $T_S^{\mu\nu}$ is locally integrable in space. Then, assuming conservation of energy-momentum in the weak sense of*

$$\int_{\partial\Omega} T_p^{\mu\nu} N_\mu dS = -\int_{\partial\Omega} T_S^{\mu\nu} N_\mu dS \tag{3}$$

for all tubular regions Ω around the particle's world-line, yields the unique force law

$$\frac{dp^\nu}{d\tau} = -\left[n_\mu T_S^{\mu\nu}(z^0, x^1)\right]_{x^1=z^1}, \tag{4}$$

where n_μ is the space-like unit covector annihilated by u^μ, and $[\cdot]_{x^1=z^1}$ denotes the jump in space at z^1.

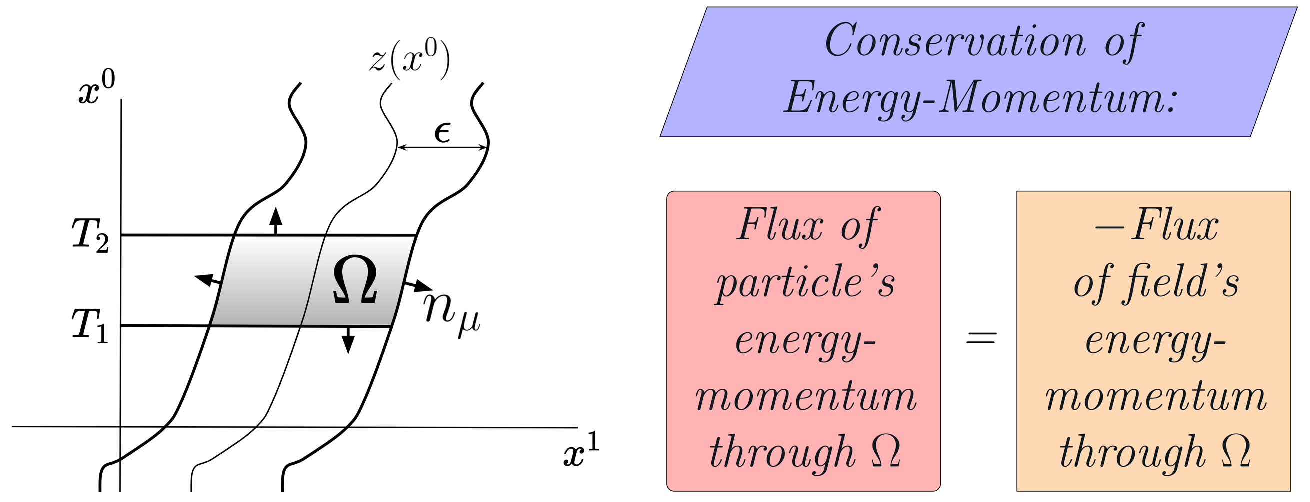

Fig. 2: Region of integration and its normals

Sketch of proof: The flux of the particle's energy-momentum through Ω is simply the change in p^ν between times T_1 and T_2:

$$\int_{\partial\Omega} T_p^{\mu\nu} N_\mu dS = p^\nu(T_2) - p^\nu(T_1). \tag{5}$$

The particle's flux has no dependence on the width ϵ of Ω, so neither should the field's. Taking the limit of the field's energy-momentum flux as $\epsilon \to 0$ returns

$$\lim_{\epsilon\to 0}\int_{\partial\Omega} T_S^{\mu\nu} N_\mu dS = \int_{T_1}^{T_2}\left[n_\mu T_S^{\mu\nu}(z^0, x^1)\right]_{x^1=z^1} d\tau. \tag{6}$$

Asymptotic Stability of Static Solution

The Cauchy problem for the massless scalar field U is given by

$$\begin{cases} \eta^{\mu\nu}\partial_\mu\partial_\nu U = a\int \delta^{(2)}(x - z(\tau))d\tau \\ U(0,x^1) = -\frac{a}{2}|x^1| + V_0(x^1) \\ \partial_0 U(0,x^1) = V_1(x^1) \end{cases} \tag{7}$$

where V_0 and V_1 are compactly supported away from the origin, and represent the incoming radiation. Using integral equation solutions for U, we explicitly calculate the force

$$\frac{dp^1}{d\tau} = F^1_{\text{self}} + F^1_{\text{ext}}. \tag{8}$$

The force resulting from the external radiation agrees with the action principle's prediction

$$F^1_{\text{ext}} = -a\partial^1 V. \tag{9}$$

However, the singular "self-force" term has been determined rather than ignored. The self-force is purely restorative, it seeks to return the point charge to rest.

$$F^1_{\text{self}} = -\frac{a^2}{2}u^1. \tag{10}$$

Theorem 2. *For any set of particle parameters $\{\tilde{m} > 0, a \in \mathbb{R}\setminus\{0\}\}$, and for any set of sufficiently small smooth functions $V_0(x^1)$, $V_1(x^1)$ compactly supported away from $x^1 = 0$, the joint initial value problem admits a unique global-in-time solution. Also, this solution asymptotically approaches another static solution.*

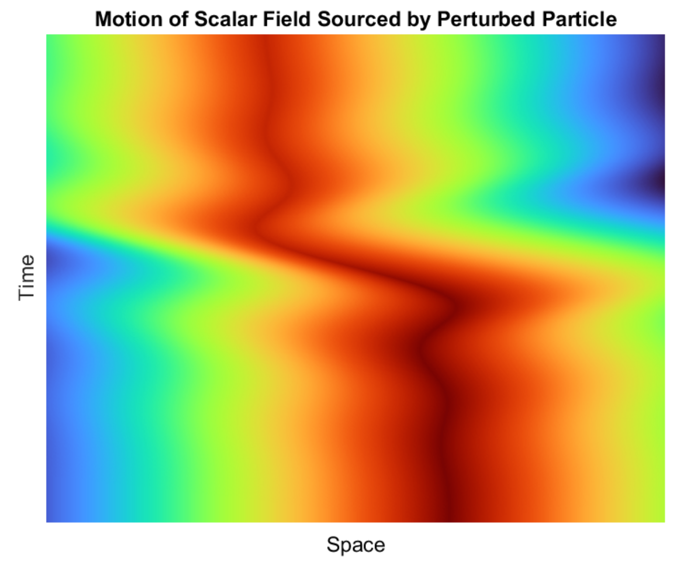

Fig. 3: Motion of scalar field generated by perturbed point charge (scan QR code on phone camera for animation).

Fig. 4: Scan for arXiv pre-print with references.