Joint evolution of a Lorentz-covariant scalar field and its point charge in one space dimension

Presented by Lawrence Frolov.
Based on work with A. Shadi Tahvildar-Zadeh and Samuel Leigh.

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What is a Classical Point Charge?

- On $n+1$ dimensional space-time $\mathbb{R}^{1,n} \cong \mathbb{R} \times \mathbb{R}^n$ with metric $\eta = \text{diag}(1, -1, -1...)$ and coordinates $x = (x^0, x^1, x^2, ...)$.

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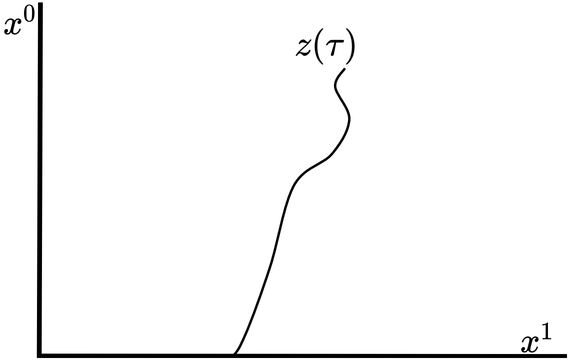
- Notation: Jump and Average:

$$\overline{f}(z) = \lim_{\epsilon \to 0} \frac{f(z+\epsilon) + f(z-\epsilon)}{2}, \quad [f]_{x=z} = \lim_{\epsilon \to 0^+} \left(f(z+\epsilon) - f(z-\epsilon)\right) \tag{1}$$

Scalar Point Charge

The scalar point charge acts as a source for the scalar field via

$$\Box U := \partial_0^2 U - \Delta U = a \int \delta^{(n+1)}(x - z(\tau))d\tau \qquad (2)$$



Motion of Scalar Field Animation

Given initial data (V_0, V_1) and a charge trajectory $z(\tau)$, the solution for U is well-known.

$$\begin{cases} \Box U = a \int \delta^{(n+1)}(x - z(\tau)) d\tau \\ U\big|_{x^0=0} = V_0 \\ \partial_0 U\big|_{x^0=0} = V_1 \end{cases} \tag{3}$$

Figure: Animation of scalar field sourced by a moving point charge in one space dimension.

(In)admissible Force Laws

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- Modify "vacuum law" to regularize sourced fields:

$$(1 + \frac{1}{\kappa^2}\Box)\Box U = J[z] \qquad \text{K-T.Z[2], Hoang et al.[3]} \tag{8}$$

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- In one space dimension, no modification is necessary!

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For initial data smooth away from $z(0)$*, this force can be explicitly computed as*

$$\frac{dp^\nu}{d\tau} = -a\overline{\partial^\nu U}. \tag{12}$$

Derivation of Force Law 1

$$\int_{\Omega} \partial_\mu T^{\mu\nu} dV = \int_{\partial\Omega} T^{\mu\nu} N_\mu dS \tag{13}$$

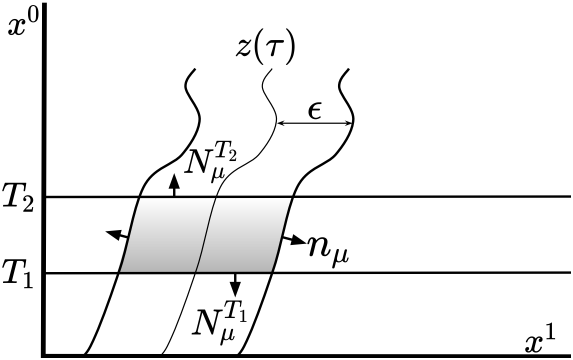
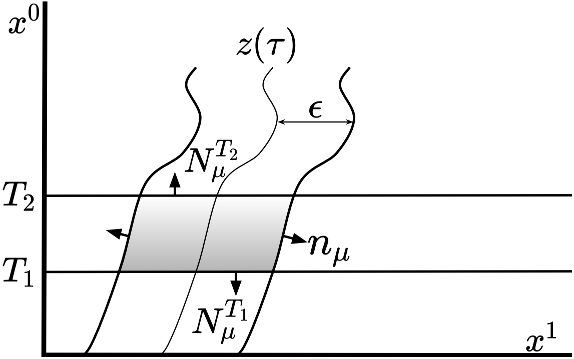


Figure: Region of integration Ω and its normals

Derivation of Force Law 1

$$\int_{\partial\Omega} T_p^{\mu\nu} N_\mu dS = p^\nu(\tau_2) - p^\nu(\tau_1) = \int_{\tau_1}^{\tau_2} \frac{dp^\nu}{d\tau} d\tau. \tag{13}$$



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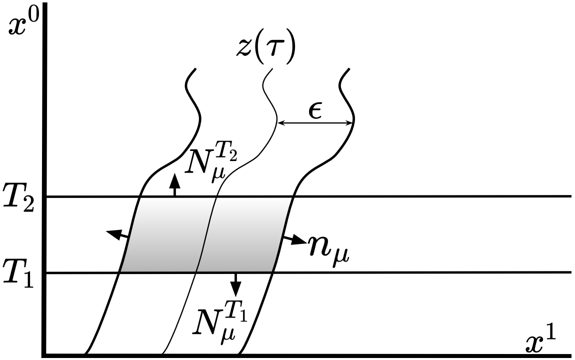

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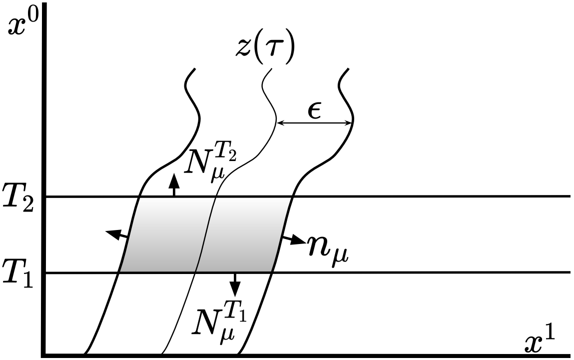
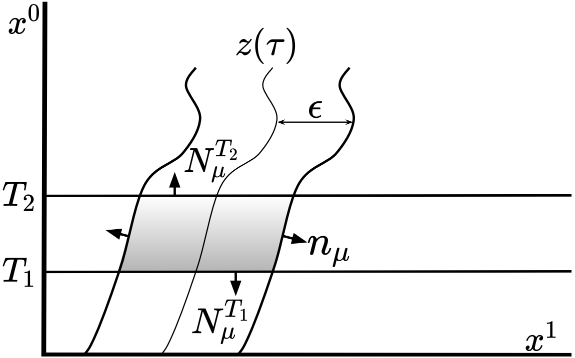


Derivation of Force Law 2

Lemma

Suppose U satisfies equation (3) with $(U, \partial_0 U)\big|_{x^0=0} \in C^{0,1}(\mathbb{R}) \times L^\infty(\mathbb{R})$ and singular source concentrated on a time-like world-line $z(\tau)$. Then U is of regularity $C^{0,1}(\mathbb{R}^{1,1})$, and $\partial_\mu U$, $T_U^{\mu\nu} \in L^\infty(\mathbb{R}^{1,1})$.



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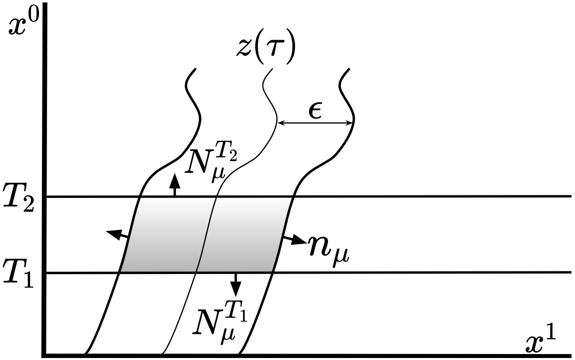
$$\lim_{\epsilon \to 0} \int_{\partial\Omega} T_U^{\mu\nu} N_\mu dS = \int_{\tau_1}^{\tau_2} \left[T_U^{\mu\nu} n_\mu(z^0, x^1) \right]_{x^1 = z^1(\tau)} d\tau \qquad (14)$$



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- $n_\mu T^{\mu\nu}$ depends on both $\partial^0 U, \partial^1 U$, which seems undesireable since we'd like $\frac{dp^\nu}{d\tau} \propto -a\partial^\nu U$.

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- $n_\mu T^{\mu\nu}$ depends on both $\partial^0 U, \partial^1 U$, which seems undesireable since we'd like $\frac{dp^\nu}{d\tau} \propto -a\partial^\nu U$.
- If we assume $(U, \partial_0 U)\big|_{x^0=0}$ is smooth away from $z^1(0)$. We then have the jump formula

$$[\partial_\mu U]_{x^1=z^1(\tau)} = -a n_\mu \tag{16}$$

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The final force law $\frac{dp^\nu}{d\tau} = -a\overline{\partial^\nu U}$ follows from

$$\left[\frac{f^2(x^1)}{2}\right]_{x^1=z^1} = [f]_{x^1=z^1}\,\overline{f} \tag{20}$$

Force Law Consequences 1

Corollary

The force contribution of external scalar fields follows the standard law derived from the principle of least action.

$$F^{\nu}_{ext} = -a\partial^{\nu} U_{ext}. \tag{21}$$

However, the singular "self-force" can now be determined, and the expression for it guarantees the conservation of the system's total energy-momentum.

$$F^{\nu}_{self} = -a\overline{\partial^{\nu} U}_{source} = \frac{-a^2 u^{\nu}}{2} \tag{22}$$

Force Law Consequences 2

Theorem

For any set of particle parameters with positive bare mass and non-zero real scalar charge, and for any "admissible" initial data, the joint initial value problem given by

$$\begin{cases} \eta^{\mu\nu}\partial_\mu\partial_\nu U(x) & = & a\int \delta^{(2)}(x - z(\tau))d\tau \\ U(0,x^1) & = & V_0(x^1) \\ \partial_0 U(0,x^1) & = & V_1(x^1), \end{cases} \tag{23}$$

$$\begin{cases} \frac{dz^\mu}{d\tau} = u^\mu \\ \frac{dp^\mu}{d\tau} = -\left[n_\mu T_U^{\mu\nu}(z^0,x^1)\right]_{x^1=z^1(\tau)}. \end{cases} \tag{24}$$

admits a unique, global-in-time solution.

Summary

- We studied the joint evolution equations for a scalar field and its point charge source in 1 space dimension

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- We studied the joint evolution equations for a scalar field and its point charge source in 1 space dimension
- We rigorously derived an equation of motion for the point charge from weak energy-momentum conservation
- The force law closely resembles the one derived from PoLA, and returns a simple expression for the self-force.

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- We rigorously derived an equation of motion for the point charge from weak energy-momentum conservation
- The force law closely resembles the one derived from PoLA, and returns a simple expression for the self-force.
- This force law allowed us to prove well-posedness of the joint evolution problem for the field-particle singularity system.

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- Point mass acts as singularity in curvature

$$g_{\mu\nu} \in C^{0,1}(\mathcal{M}), \quad \left[\Gamma^{\kappa}_{\mu\nu}(z^0, x^1)\right]_{x^1=z^1} \neq 0 \tag{25}$$

$$\nabla_\mu T_p^{\mu\nu} \propto \partial_\mu T_p^{\mu\nu} + \Gamma T_p = 0? \tag{26}$$

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- In 1 space dimension, a non-linear theory of distributions will be necessary to study the gravitational joint evolution problem.
- In higher dimensions we need to study non-linear wave equations coupled to point sources.
- Non-linear modifications may regularize the sourced fields to permit a well-posed joint evolution

$$\partial_\mu \left(\frac{\partial^\mu U}{\sqrt{1 + \partial_\alpha U \partial^\alpha U}} \right) = a \int \delta^{(n+1)}(x - z(\tau)) d\tau. \qquad (25)$$

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White Lies

- True force law from Energy-Momentum conservation:

$$\int_{\tau_1}^{\tau_2} \frac{dp^\nu}{d\tau} d\tau = \lim_{\epsilon \to 0^+} \left(\int_{\tau_1}^{\tau_2} n_\mu T_U^{\mu\nu}(z^0, z^1 + \epsilon) - n_\mu T_U^{\mu\nu}(z^0, z^1 - \epsilon) d\tau \right) \tag{26}$$

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- The momentum of a scalar point charge is $p^\mu := (m - aU(z))u^\mu \neq mu^\mu$
- Action Principle:

$$S[U, z] = \int (\mathcal{L}_U + \mathcal{L}_i + \mathcal{L}_p)\sqrt{-\eta}dx^2 \tag{27}$$

$$\mathcal{L}_U := \frac{1}{2}\eta^{\mu\nu}\partial_\mu U \partial_\nu U, \qquad \mathcal{L}_i := \frac{-a}{m}U(z)\mathcal{L}_p \tag{28}$$

$$\mathcal{L}_p := -\frac{m}{\sqrt{-\eta}} \int \sqrt{\eta_{\mu\nu}\dot{z}^\mu \dot{z}^\nu}\delta^{(2)}(x - z(\theta))d\theta \tag{29}$$