Schrödinger operators that model hard detection

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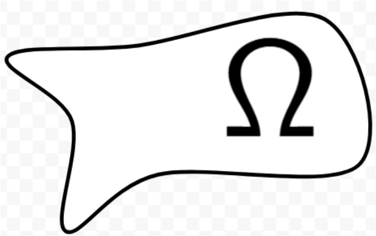
A simple question

- A non-relativistic quantum particle is prepared with state ψ_0 at $t = 0$ inside some bounded region Ω, and detectors are placed along the boundary $\partial\Omega$.



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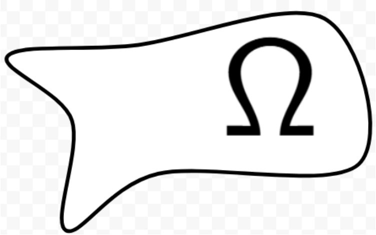

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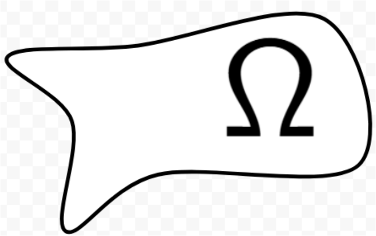

- The quantum particle evolves in Ω until it is detected along $\partial\Omega$, we record the time and position of detection.
- As the experiment is repeated: what is the distribution of times that the particle is detected along $\partial\Omega$?

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- Allcock: For hard detector, take $v \to \infty$. However, this returns unitary dynamics for ψ, with $||\psi_t||^2_{L^2(\Omega)} = 1$ for all time. The particle is never detected along $\partial\Omega$!

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- Ψ_t satisfies a norm-preserving Schrödinger evolution of the form

$$i\partial_t \Psi = \hat{H}_S \Psi$$

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Derivation 2: Hard Detectors

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The projected particle-detector wave function remains a pure-product state $\Psi_t\big|_{\mathcal{H}_P} = \psi_t \otimes \phi_t$*. The dynamics of the quantum particle in* Ω *before detection is given by the wave function* ψ_t*.*

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Remark

The wave function of the particle-detector system after detection, i.e $\Psi_t\big|_{\mathcal{H}_F}$ *is allowed to be (and will most certainly be) entangled.*

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Condition (C1)

ψ_t *weakly satisfies a Schrödinger equation inside* Ω*.*

$$i\frac{\partial \psi}{\partial t} = \hat{H}^* \psi \quad \text{in } \Omega \tag{3}$$

Where $\hat{H} = -\Delta + V$ *is defined on* $D(\hat{H}) = C_c^\infty(\Omega)$ *with* $V \in L^\infty(\Omega)$ *a real valued potential depending on the experimental apparatus.*

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- *The dynamics of $\Psi_t\big|_{\mathcal{H}_P} = \psi_t \otimes \phi_t$ are norm-non-increasing and autonomous, they are not affected by the dynamics of $\Psi_t\big|_{\mathcal{H}_F}$.*

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For fixed ϕ_0, *the evolution maps* $W_t : \psi_0 \mapsto \psi_t$, *defined for* $t \geq 0$ *forms a strongly continuous semigroup on* $L^2(\Omega)$*:*

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2. *The maps W_t are linear.*
3. *They are strongly continuous, $\lim_{t \to t_0} ||W_t \psi - W_{t_0} \psi||_{L^2(\Omega)} = 0$ for all $\psi \in L^2(\Omega)$, $t_0 \geq 0$.*

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Remark

We hope that W_t *does not depend on the fine details of the quantum state* ϕ_0*, as it is not experimentally feasible to fine-tune the initial state of a macroscopic object!*

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Condition (C3)

W_t are contractions, i.e $||W_t \psi||_{L^2(\Omega)} \leq ||\psi||_{L^2(\Omega)}$ for all $\psi \in L^2(\Omega)$.

Absorbing Boundary Conditions

- R. Tumulka proposed that hard detection should be modeled by a time-independent *local absorbing boundary condition*.

Absorbing Boundary Conditions

- He argues that ψ_t should be governed by an IBVP

$$\begin{cases} i\partial_t\psi = (-\Delta + V)\psi & \text{in } \Omega \\ \psi = \psi_0 & \text{at } t = 0 \\ \partial_n\psi = i\beta\psi & \text{on } \partial\Omega \end{cases} \tag{4}$$

where ∂_n denotes the outwards normal derivative of Ω, and β is a function on $\partial\Omega$ satisfying $\text{Re}(\beta) \geq 0$.

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Proposal (Tumulka's Absorbing Boundary Rule)

For ψ_t satisfying (4) with $||\psi_0||_{L^2(\Omega)} = 1$, the probability of detecting the quantum particle in $\Sigma \subset \partial\Omega$ between times t_1 and t_2 is

$$Prob_{\psi_0}(t_1 \leq t \leq t_2, x \in \Sigma) = \int_{t_1}^{t_2}\int_{\Sigma} \vec{n} \cdot \vec{j}_{\psi_t}\, dx^{n-1} dt \quad (5)$$

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Theorem (L.F, S.Teufel, R. Tumulka)

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain of dimension $n > 1$ and $\beta \in C^1(\partial\Omega)$ with $Re(\beta) \geq 0$ a.e. Then for $\psi_0 \in H^2(\Omega)$ initially satisfying the boundary condition $\partial_n\psi_0 = i\beta\psi_0$ on $\partial\Omega$, there exists a unique global-in-time solution to (4).

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In addition, the solution maps $W_t : \psi_0 \mapsto \psi_t$ extend to a C_0 contraction semigroup on $L^2(\Omega)$.

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- $-iL$ *is maximally dissipative; i.e* $Re\langle -iL\psi, \psi\rangle_{\mathcal{H}} \leq 0$ *for all* $\psi \in D(L)$; *and* $-iL$ *has no dissipative extensions.*

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The converse is also true, if $-iL$ is densely defined and maximally dissipative on $\mathcal{H}$ then it generates a C_0 contraction semigroup.

Intuition

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- W_t "weakly solving Schrödinger equation": $W_t = \exp(-itL)$ with $L \subset \hat{H}^*$, where $\hat{H} = (-\Delta + V)\big|_{C_c^\infty(\Omega)}$.
- If $-iL = -i\hat{H}^*\big|\{\psi \in D(\hat{H}^*) : \psi \text{ satisfies some B.C}\}$ is maximally dissipative, then Lumer-Phillips says that for $\psi_0 \in D(L)$

$$\begin{cases} i\partial_t\psi = \hat{H}^*\psi & \text{in } \Omega \\ \psi = \psi_0 & \text{at } t = 0 \\ \text{B.C} & \text{on } \partial\Omega \end{cases} \tag{5}$$

has a unique global-in-time solution $\psi_t = \exp(-itL)\psi_0 \in D(L)$, and the solution mappings extend continuously to $L^2(\Omega)$.

Parameterization of ABC in 1-D

Theorem ((Informal))

Let $\Omega = (-\infty, 0]$, $V \in L^\infty(\Omega, \mathbb{R})$, *and* $\hat{H} = (-\partial_x^2 + V)\big|_{C_c^\infty(\Omega)}$.

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$$\begin{cases} i\partial_t\psi = (-i\partial_x^2 + V)\psi & \text{in } \Omega \\ \psi = \psi_0 & \text{at } t = 0 \\ \psi(0) + i\partial_x\psi(0) = \Phi(\psi(0) - i\partial_x\psi(0)) & \text{at } x = 0 \end{cases} \tag{6}$$

where $\Phi \in \mathbb{C}$ *satisfies* $|\Phi| \leq 1$.

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where $\Phi \in \mathbb{C}$ satisfies $|\Phi| \leq 1$.W_t is unitary if and only if $|\Phi| = 1$, otherwise W_t is norm-decreasing.

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$$D(L_\Phi) = \{\psi \in H^2(\Omega) : \psi(0) + i\partial_x\psi(0) = \Phi(\psi(0) - i\partial_x\psi(0))\} \qquad (7)$$

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where $\Phi \in \mathbb{C}$ *satisfies* $|\Phi| \leq 1.L_\Phi$ *is self-adjoint if and only if* $|\Phi| = 1$, *otherwise* $\exp(-itL)$ *is norm-decreasing.*

Parameterization of ABC in 1-D

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Let $\Omega = (-\infty, 0]$, $V \in L^\infty(\Omega, \mathbb{R})$, *and* $\hat{H} = (-\partial_x^2 + V)\big|_{C_c^\infty(\Omega)}$. *Then* $-iL$ *is a maximally dissipative restriction of* $-i\hat{H}^*$ *if and only if it is of the form*

$$D(L_\Phi) = \{\psi \in H^2(\Omega) : \psi(0) + i\partial_x\psi(0) = \Phi(\psi(0) - i\partial_x\psi(0))\} \tag{7}$$

where $\Phi \in \mathbb{C}$ *satisfies* $|\Phi| \leq 1$.L_Φ *is self-adjoint if and only if* $|\Phi| = 1$, *otherwise* $\exp(-itL)$ *is norm-decreasing.*

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$$(1 + \Phi)\partial_x \psi(0) = i(1 - \Phi)\psi(0)$$

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- Neumann B.C: $\Phi = 1$ implies $D(L_1) = \{\psi \in H^2(\Omega) : \partial_x \psi(0) = 0\}$
- $|\Phi| < 1$ implies $D(L_\Phi) = \{\psi \in H^2(\Omega) : \partial_x \psi(0) = i\frac{1-\Phi}{1+\Phi}\psi(0)\}$

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$$2\mathrm{Re}\langle -i\hat{H}^*\psi, \psi\rangle_{L^2((-\infty])} = J_\psi(0) = ||G_+\psi||_{\mathbb{C}}^2 - ||G_-\psi||_{\mathbb{C}}^2 \quad (8)$$

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- Taking $-iL \subset -i\hat{H}^*$ maximally dissipative implies

$$||G_+\psi||^2_{\mathbb{C}} - ||G_-\psi||^2_{\mathbb{C}} = 2\text{Re}\langle -iL\psi, \psi\rangle \leq 0, \quad \forall \psi \in D(L) \quad (9)$$

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- Hence $|G_+\psi| \leq |G_-\psi|$ for all $\psi \in D(L)$, in particular $G_-\psi$ uniquely determines $G_+\psi$.

Main Results 1

Theorem (Absorbing Boundary Condition Derivation)

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, and let W_t be a C_0 contraction semigroup on $L^2(\Omega)$ weakly solving $i\partial_t\psi = \hat{H}^*\psi$.

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decomposes the probability leaving Ω into a difference between the squares of two norms

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Associated to the evolution operator W_t there exists a unique linear contraction $\Phi : L^2(\partial\Omega) \to L^2(\partial\Omega)$ such that for all $\psi_0 \in L^2(\Omega)$, $W_t\psi_0$ uniquely solves the initial-boundary value problem

$$\begin{cases} i\partial_t\psi = \hat{H}\psi & \text{in } \Omega \\ \psi = \psi_0 & \text{at } t = 0 \\ G_+\psi = \Phi G_-\psi & \text{on } \partial\Omega \end{cases} \tag{11}$$

Main Results 2

Remark

Most linear contractions Φ result in boundary conditions with highly non-local dynamics. This is not surprising, given that conditions (C1), (C2), and (C3) do not rule out cases where probability is instantly transported from one part of the boundary to another.

Main Results 2

Theorem (Robin Boundary Condition)

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, and let $\beta \in L^\infty(\partial\Omega)$ satisfy $Re(\beta) \geq 0$ a.e on $\partial\Omega$. Then the initial-boundary value problem

$$\begin{cases} i\partial_t \psi = \hat{H}\psi & \text{in } \Omega \\ \psi = \psi_0 & \text{at } t = 0 \\ \partial_n \psi = i\beta\psi & \text{on } \partial\Omega \end{cases} \tag{12}$$

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We can prove this by explicitly constructing a contraction Φ_β such that

$$G_+\psi = \Phi_B G_-\psi \iff \partial_n\psi\big|_{\partial\Omega} = i\beta\psi\big|_{\partial\Omega} \tag{13}$$

Main Results 3

- (Energy-Time Uncertainty) For $\psi \in C_c^\infty(\Omega)$ and Φ linear contraction on $L^2(\partial\Omega)$, set $p = 1 - \lim_{t\to\infty} || \exp(-it\hat{H}_\Phi)\psi||$. Then

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- (Continuity in β): If β_ϵ is a sequence converging to β the detection time distributions converge

$$2\mathrm{Re}(\beta_\epsilon)\left|\exp(-it\hat{H}_{\beta_\epsilon})\psi_0\right|^2\Big|_{\partial\Omega} \xrightarrow{\epsilon\to 0} 2\mathrm{Re}(\beta)\left|\exp(-it\hat{H}_\beta)\psi_0\right|^2\Big|_{\partial\Omega} \tag{15}$$

with respect to $L^1_{\mathrm{Loc}}([0,\infty)\times\partial\Omega)$ norm.

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- Any such W_t gives rise to a well-defined distribution of detection times along $\partial\Omega$.
- We showed that these distributions are stable under small perturbations.

Outlook

- For the $N-$particle case $\psi_t \in L^2(\Omega^N)$, hence we would like to generalize this result to bounded Lipschitz regions.

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- Relativistic setting: Parameterize all maximally dissipative restrictions of $\hat{H}^*_{\text{Dirac}}$
- Which ABC's for the non-relativistic Schrödinger equation are derivable as a non-relativistic limit of the Dirac equation with some ABC?

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