Irreversible hard detection of non-relativistic quantum particles

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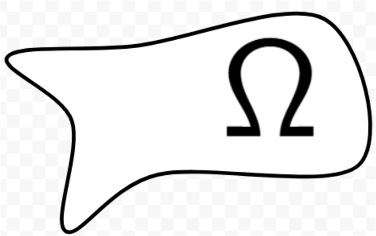
A simple question

- A non-relativistic quantum particle is prepared with state ψ_0 at $t = 0$ inside some bounded region Ω, and detectors are placed along the boundary $\partial\Omega$.



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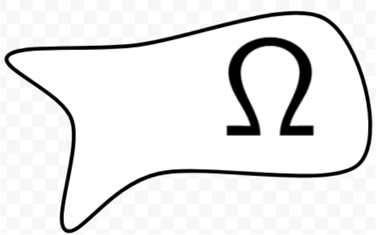

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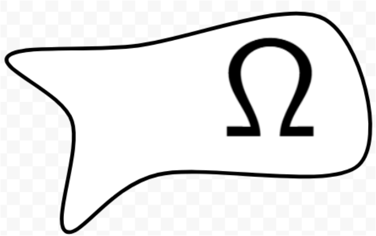

- The quantum particle freely evolves in Ω until it is detected along $\partial\Omega$, we record the time and position of detection.
- As the experiment is repeated: what is the distribution of times that the particle is detected along $\partial\Omega$?

Quantum Mechanics

- Quantum system of N non-relativistic particles: the wave function $\psi_t \in L^2(\mathbb{R}^{3N})$ evolves according to the Schrödinger E.Q.

$$i\hbar\partial_t\psi = \left(\sum_{j=1}^{N} \frac{-\hbar^2}{2m_j}\Delta_j \quad + \quad V(\mathbf{x})\right)\psi = \hat{H}\psi \tag{1}$$

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Law (Born Rule)

For a measurable set $D \subseteq \mathbb{R}^{3N}$, the probability that N particles with initial wave function ψ_0 have positions $(x_1, x_2, \ldots, x_N) \in D$ at time t is

$$Prob((x_1, x_2, \ldots, x_N) \in D)[\psi_0] = \int_D \rho(t, \mathbf{x})\ d^{3N}\mathbf{x} = \int_D |\psi_t|^2(\mathbf{x})\ d^{3N}\mathbf{x} \tag{2}$$

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- The Schrödinger equation tells us how these probabilities are changing in time.

$$\frac{d}{dt}\mathrm{Prob}\left((x_1, x_2, \ldots, x_N) \in D\right)[\psi_0] = -\int_{\partial D} \vec{j}_{\psi_t} \cdot \vec{n}\, dS \qquad (2)$$

- where $\vec{n}$ denotes the unit normal vector to $\partial\Omega$, dS the surface element along ∂D, and $\vec{j}_\psi := \frac{\hbar}{m}\Im(\psi^*\vec{\nabla}\psi)$

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- But we have data!

Double Slit Experiment

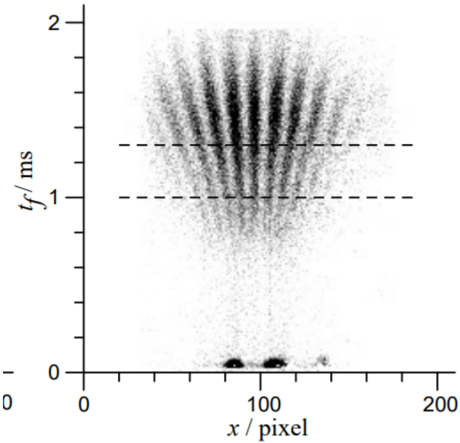

Figure: Kurtsiefer, Pfau, and Mlynek.

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$$\text{Prob}_{\text{B.M}}(t_d \in [t_1, t_2])[\psi_0] \approx \int_{t_1}^{t_2} \int_{\partial\Omega} \vec{j}_{\psi_t} \cdot \vec{n} \;\; dS \;\; dt \tag{4}$$

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- The theory of boundary tuples is **necessary** for this derivation!

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- Let $\mathcal{N} \subset \mathbb{R}^{3N}_D$ denote set of detector configurations in which detectors have not yet clicked but are ready, and $\mathcal{F} \subset \mathbb{R}^{3N}_D$ denote configurations in which a detector has fired.

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- We take the support of ψ_0 to be contained in Ω, and the support of ϕ_0 to be contained in $\mathcal{N}$.

Derivation 2: Hard Detectors

Assumption (Hard Detector)

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Remark

The wave function of the particle-detector system after detection, i.e $\Psi_t\big|_{\mathbb{R}^3_p\times\mathcal{F}}$ is allowed to be (and will most certainly be) entangled.

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Condition (C0)

The particle-detector system remains a pure-product state inside $\Omega \times \mathcal{N}$, i.e $\Psi_t\big|_{\Omega\times\mathcal{N}} = \psi_t \otimes \phi_t$. The dynamics of the quantum particle in Ω before detection is given by the wave function ψ_t.

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Condition (C1)

ψ_t *weakly satisfies a Schrödinger equation inside* Ω*.*

$$i\frac{\partial\psi}{\partial t} = \hat{H}^*\psi \quad \text{in } \Omega \tag{6}$$

Where $\hat{H} = -\Delta + V$ *is defined on* $D(\hat{H}) = H_0^2(\Omega)$ *with* $V \in L^\infty(\Omega)$ *a real valued potential depending on the experimental apparatus.*

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- It follows from autonomy that for any fixed initial detector state ϕ_0, the evolution mapping $W_t : \psi_0 \mapsto \psi_t$ where ψ_t uniquely satisfies

$$\begin{cases} i\partial_t(\psi_t \otimes \phi_t) &= \hat{\mathcal{H}}_S(\psi_t \otimes \phi_t) \quad \text{in } \Omega \times \mathcal{N} \\ (\psi_t \otimes \phi_t)\Big|_{t=0} &= \psi_0 \otimes \phi_0 \end{cases} \tag{7}$$

is well defined.

Evolution Maps

Condition (C2)

The evolution maps $W_t : L^2(\Omega) \to L^2(\Omega)$, $\psi_t := W_t \psi_0$, *defined for* $t \geq 0$ *form a* C_0 *semigroup:*

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1. *The maps* W_t *are linear.*
2. *They are strongly continuous,* $\lim_{t\to t_0} ||W_t\psi - W_{t_0}\psi||_{L^2(\Omega)} = 0$ *for all* $\psi \in L^2(\Omega)$, $t_0 \geq 0$. *(because evolution of* Ψ_t *and* ϕ_t *is continuous)*

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3. *They form a semigroup under composition, i.e* $W_t W_s = W_{t+s}$ *for* $t, s \geq 0$, *with* $W_0 = \mathbb{1}$. *(by autonomy)*

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Remark

We hope that W_t *does not depend on the fine details of the quantum state* ϕ_0, *as it is not experimentally feasible to fine-tune the initial state of a macroscopic object!*

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Condition (C3)

W_t *are contractions, i.e* $||W_t\psi||_{L^2(\Omega)} \leq ||\psi||_{L^2(\Omega)}$ *for all* $\psi \in L^2(\Omega)$.

Absorbing Boundary Conditions

- Roderich(Rodi) Tumulka argued that hard detection should be modeled by a time-independent *local absorbing boundary condition*.

Absorbing Boundary Conditions

- He argues that ψ_t should be governed by an IBVP

$$\begin{cases} i\partial_t\psi = (-\Delta + V)\psi & \text{in } \Omega \\ \psi = \psi_0 & \text{at } t = 0 \\ \partial_n\psi = iB\psi & \text{on } \partial\Omega \end{cases} \tag{8}$$

where ∂_n denotes the outwards normal derivative of Ω, and B is a function on $\partial\Omega$ satisfying $\mathrm{Re}(B) \geq 0$.

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Proposal (Tumulka's Absorbing Boundary Rule)

For ψ_t satisfying (8) with $||\psi_0||_{L^2(\Omega)} = 1$, the probability of detecting the quantum particle in $B \subset \partial\Omega$ between times t_1 and t_2 is

$$Prob_{\psi_0}(t_1 \leq t \leq t_2, x \in \Sigma) = \int_{t_1}^{t_2} \int_{\Sigma} \vec{n} \cdot \vec{j}_{\psi_t} \, dx^{n-1} dt \tag{9}$$

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Main Results 1

Theorem (Absorbing Boundary Condition Derivation)

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, and let W_t be a C_0 contraction semigroup on $L^2(\Omega)$ weakly solving $i\partial_t\psi = \hat{H}^*\psi$.

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$$G_\pm\psi := \frac{1}{\sqrt{2}}\left(\iota_-\psi\big|_{\partial\Omega} \pm i\iota_+\partial_n\psi_D\big|_{\partial\Omega}\right) \tag{10}$$

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$$G_\pm\psi := \frac{1}{\sqrt{2}}\left(\iota_-\psi\big|_{\partial\Omega} \pm i\iota_+\partial_n\psi_D\big|_{\partial\Omega}\right) \tag{10}$$

decomposes the probability leaving Ω into a difference between the squares of two norms

$$\frac{d}{dt}||W_t\psi||^2_{L^2(\Omega)} = ||G_+\psi||^2_{L^2(\partial\Omega)} - ||G_-\psi||^2_{L^2(\partial\Omega)}. \tag{11}$$

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Theorem (Absorbing Boundary Condition Derivation)

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, and let W_t be a C_0 contraction semigroup on $L^2(\Omega)$ weakly solving $i\partial_t\psi = \hat{H}^*\psi$. First, the maps $G_\pm : D(\hat{H}^*) \to L^2(\partial\Omega)$ defined linearly in $(\psi, \partial_n\psi)\big|_{\partial\Omega}$ decomposes the probability leaving Ω into a difference between the squares of two norms

$$\frac{d}{dt}||W_t\psi||^2_{L^2(\Omega)} = ||G_+\psi||^2_{L^2(\partial\Omega)} - ||G_-\psi||^2_{L^2(\partial\Omega)}. \tag{10}$$

Associated to the evolution operator W_t there exists a unique linear contraction $\Phi : L^2(\partial\Omega) \to L^2(\partial\Omega)$ such that for all $\psi_0 \in L^2(\Omega)$, $W_t\psi_0$ uniquely solves the initial-boundary value problem

$$\begin{cases} i\partial_t\psi = \hat{H}\psi & \text{in } \Omega \\ \psi = \psi_0 & \text{at } t = 0 \\ G_+\psi = \Phi G_-\psi & \text{on } \partial\Omega \end{cases} \tag{11}$$

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Stated rigorously, there exists a unique linear contraction Φ such that $W_t = e^{-it\hat{H}_\Phi}$, where $\hat{H}_\Phi$ is the closed extension of $\hat{H}$ defined by

$$D(\hat{H}_\Phi) := \{\psi \in D(\hat{H}^*) : G_+\psi = \Phi G_-\psi\}, \quad \hat{H}_\Phi := \hat{H}^*\big|_{D(\hat{H}_\Phi)}. \tag{11}$$

The converse is also true, any linear contraction $\Phi : L^2(\partial\Omega) \to L^2(\partial\Omega)$ gives rise to a unique C_0 contraction semigroup $e^{-it\hat{H}_\Phi}$.

Main Results 2

Remark

Most linear contractions Φ result in boundary conditions with highly non-local dynamics. This is not surprising, given that conditions (C1), (C2), and (C3) do not rule out cases where probability is instantly transported from one part of the boundary to another.

Main Results 2

Theorem (Robin Boundary Condition)

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, and let $B : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ be a compact operator such that $\mathrm{Re}\langle B\chi, \chi\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} \geq 0$ for all $\chi \in H^{1/2}(\partial\Omega)$. Then the initial-boundary value problem

$$\begin{cases} i\partial_t\psi = \hat{H}\psi & \text{in } \Omega \\ \psi = \psi_0 & \text{at } t = 0 \\ \partial_n\psi = iB\psi & \text{on } \partial\Omega \end{cases} \tag{12}$$

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admits a unique, global-in-time solution for each $\psi_0 \in L^2(\Omega)$.

We can prove this by explicitly constructing a contraction Φ_B such that

$$G_+\psi = \Phi_B G_-\psi \iff \partial_n\psi\big|_{\partial\Omega} = iB\psi\big|_{\partial\Omega} \tag{13}$$

Main Results 3

- (Energy-Time Uncertainty) For $\psi \in C_c^\infty(\Omega)$and Φ contraction on $L^2(\partial\Omega)$, let $p = 1 - \lim_{t\to\infty} ||e^{-it\hat{H}_\Phi}\psi||$. Then

$$\sigma_{\hat{H}_\Phi,\psi}\sigma_{T,\psi} \geq \frac{p}{2}. \tag{14}$$

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- (Continuity in Φ) Let Φ_ϵ be a family of linear contractions on $L^2(\partial\Omega)$. Then

$$\exp(-it\hat{H}_{\Phi_\epsilon})\psi_0 \xrightarrow{\epsilon\to 0} \exp(-it\hat{H}_\Phi)\psi_0 \tag{15}$$

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- Detection time distributions also depend continuously on Φ.

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- We showed that these distributions are stable under small perturbations of Φ.

Outlook

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- Tumulka's proposal: For an *ideal* detector most sensitive to particles with energy $\kappa > 0$, we set the boundary condition as $\partial_{\boldsymbol{n}}\psi = i\kappa\psi$.
- Perhaps a better understanding of the spectrum of $\hat{H}$ with boundary condition $\partial_{\boldsymbol{n}}\psi = iB\psi$ will guide us to the right answer.

References

- W. Pauli, in: S. Flugge (Ed.), Encyclopedia of Physics, Vol. 5/1, Springer, Berlin, 1958, p. 60.

References

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References

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- J. Kijowski, Rep. Math. Phys. 6 (1974) 362.

References

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- J. Kijowski, Rep. Math. Phys. 6 (1974) 362.
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References

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- R. Werner, Ann. Inst. Henri Poincare 47 (1987) 429.

References

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- R. Werner, Ann. Inst. Henri Poincare 47 (1987) 429.
- Jukka Kiukas et al 2012 J. Phys. A: Math. Theor. 45 185301

References

- R. Tumulka: Distribution of the Time at Which an Ideal Detector Clicks. Annals of Physics 442: 168910 (2022) http://arxiv.org/abs/1601.03715

References

- R. Tumulka: Distribution of the Time at Which an Ideal Detector Clicks. Annals of Physics 442: 168910 (2022) http://arxiv.org/abs/1601.03715
- R. Tumulka: Detection Time Distribution for Several Quantum Particles. Physical Review A 106: 042220 (2022) http://arxiv.org/abs/1601.03871

References

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- R. Tumulka: Detection Time Distribution for Several Quantum Particles. Physical Review A 106: 042220 (2022) http://arxiv.org/abs/1601.03871
- R. Tumulka: Detection Time Distribution for the Dirac Equation. Preprint (2016) http://arxiv.org/abs/1601.04571

References

- R. Tumulka: Distribution of the Time at Which an Ideal Detector Clicks. Annals of Physics 442: 168910 (2022) http://arxiv.org/abs/1601.03715
- R. Tumulka: Detection Time Distribution for Several Quantum Particles. Physical Review A 106: 042220 (2022) http://arxiv.org/abs/1601.03871
- R. Tumulka: Detection Time Distribution for the Dirac Equation. Preprint (2016) http://arxiv.org/abs/1601.04571
- R. Tumulka: Absorbing Boundary Condition as Limiting Case of Imaginary Potentials. Communications in Theoretical Physics 75: 015103 (2023) http: //arxiv.org/abs/1911.12730

References

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- R. Tumulka: Detection Time Distribution for the Dirac Equation. Preprint (2016) http://arxiv.org/abs/1601.04571
- R. Tumulka: Absorbing Boundary Condition as Limiting Case of Imaginary Potentials. Communications in Theoretical Physics 75: 015103 (2023) http: //arxiv.org/abs/1911.12730
- R. Tumulka: On a Derivation of the Absorbing Boundary Rule. Physics Letters A 494: 129286 (2024) http://arxiv.org/abs/2310.01343

Werner's Arrival Time Proposal

Definition (Exit space)

For a C_0 contraction semigroup $W_t = e^{-itL}$ with densely defined generator L on a Hilbert space $\mathcal{H}$, an **exit space** for L consists of a Hilbert space $\mathcal{K}$ and a mapping $j : D(L) \to \mathcal{K}$ satisfying

$$\langle j\psi, j\phi\rangle_{\mathcal{K}} = \langle iL\psi, \phi\rangle_{\mathcal{H}} + \langle \psi, iL\phi\rangle_{\mathcal{H}} = -\frac{d}{dt}\langle W_t\psi, W_t\phi\rangle_{\mathcal{H}}\bigg|_{t=0}. \tag{16}$$

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For $\psi \in D(L)$, we define $(J\psi)(t) : \mathbb{R}_+ \to \mathcal{K}$ as $(J\psi)(t) := j(W_t\psi)$, so

$$\int_0^\infty ||J\psi||_{\mathcal{K}}^2(t)dt = -\int_0^\infty \frac{d}{dt}||W_t\psi||_{\mathcal{H}}^2 dt = ||\psi||_{\mathcal{H}}^2 - \lim_{t\to\infty} ||W_t\psi||_{\mathcal{H}}^2. \quad (17)$$

It follows that J extends to a continuous map $\mathcal{H} \to L^2(\mathbb{R}_+, \mathcal{K})$.

Werner's Arrival Time Proposal 2

- The quantity $||J\psi_0||^2_{\mathcal{K}}(t)$ is the rate at which probability flows from $\Omega \times \mathcal{N}$ to $\mathbb{R}^n \times \mathcal{F}$.

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Theorem

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$$D(\hat{H}_\Phi) := \{\psi \in D(\Delta^*) : G_+\psi = \Phi G_-\psi\}, \quad \hat{H}_\Phi := (-\Delta + V)^*\big|_{D(\hat{H}_\Phi)}. \tag{18}$$

Then an exit space for W_t can always be constructed with $\mathcal{K} = L^2(\partial\Omega)$

$$j_\Phi : D(\hat{H}_\Phi) \to L^2(\partial\Omega), \quad j_\Phi\psi := \sqrt{1 - \Phi^*\Phi}\, G_-\psi \tag{19}$$

Detection Time Proposal

Proposal (Born Rule for Detection Times)

Let $\Omega \subset \mathbb{R}^n$ *be a bounded* C^2 *domain. Prepare the quantum particle at time* 0 *with initial wave function* ψ_0 *of unit norm and supported in* Ω*.*

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$$Prob(t_D = \infty) = 1 - ||J_\Phi\psi_0||^2_{L^2((0,\infty),L^2(\partial\Omega))} = \lim_{t\to\infty} ||W_t\psi||^2_{L^2(\partial\Omega)}. \tag{21}$$

Energy - Detection Time Uncertainty Principle

Theorem (Uncertainty Principle, Kiukas et. al. 2012)

For $\psi \in D(\hat{H}_\Phi) \cap \ker(j_\Phi)$ *with unit norm, let* $p = 1 - \lim_{t\to\infty} ||W_t\psi||^2_{L^2(\Omega)}$ *denote the probability that the particle prepared in state* ψ *is ever detected.*

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along with the conditional time variance $\sigma^2_{T,\psi}$

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satisfy the inequality

$$\sigma_{\hat{H}_\Phi,\psi}\sigma_{T,\psi} \geq \frac{\sqrt{p}}{2}. \tag{24}$$

Stability of Detection Time Distributions

Theorem (Convergence of Detection Time Distributions)

Let $\Phi_\epsilon : L^2(\partial\Omega) \to L^2(\partial\Omega)$ *be a family of linear contractions with* $\Phi_\epsilon \to \Phi$ *as* $\epsilon \to 0$.

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- *The exit space wave functions $J_{\Phi_\epsilon}\psi_0$ converge as $\epsilon \to 0$ in $L^2_{Loc}((0,\infty), L^2(\partial\Omega))$ to $J_\Phi \psi_0$.*

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- *The exit space wave functions $J_{\Phi_\epsilon}\psi_0$ converge as $\epsilon \to 0$ in $L^2_{Loc}((0,\infty), L^2(\partial\Omega))$ to $J_\Phi\psi_0$.*
- *Consequently, the detection time probabilities converge as $\epsilon \to 0$ over finite time intervals*

$$\lim_{\epsilon\to 0} ||J_{\Phi_\epsilon}\psi_0||^2_{L^2([t_1,t_2],L^2(\partial\Omega))} = ||J_\Phi\psi_0||^2_{L^2([t_1,t_2],L^2(\partial\Omega))} \tag{25}$$

for all $\psi_0 \in L^2(\Omega)$ and any finite time interval $[t_1, t_2]$.