Homework 1 (due 9/16 Wed)

§0.1: Sets and Set Operations

- Textbook §0 Exercise #6
- Textbook §0 Exercise #8
- Textbook §0 Exercise #11

§0.2: Relations and Functions

- For any sets X, Y, Z, prove $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.
- Textbook §0 Exercise #14
- Let X, Y be sets. For any function $f : X \to Y$ and any subset $A \subseteq Y$, we define the *preimage of A under f* to be the subset $f^{-1}(A) := \{x \in X \mid f(x) \in A\} \subseteq X$. For any function $f : X \to Y$ and any subsets $A \subseteq Y$, $B \subseteq Y$, prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

§0.3: Mathematical Induction

- Textbook §0 Exercise #21
- Textbook §0 Exercise #23
- Textbook §0 Exercise #26
- *I was going to ask you to prove, for a well-ordered set X, the Principle of Complete Induction for a statement concerning a variable $x \in X$. However, after talking to some of you about the worksheet during the discussion section today, I am now having a second thought about this. Whatever I decide, I will assign one more problem (from §0.3, or possibly from §0.2) tonight.*

Worksheet 1

1. Let $X = \{x, y, z\}$ and $\mathbf{2} = \{0, 1\}$. Define $X^{\mathbf{2}}$ to be the set of all functions $f : \mathbf{2} \to X$, and $\mathbf{2}^X$ to be the set of all functions from $g : X \to \mathbf{2}$.

   (a1) Write down all elements (functions) of $X^{\mathbf{2}}$ explicitly; this means, for each element (function), you must specify its value in the codomain at each point in the domain.
   (a2) Find a natural bijective function from $X^{\mathbf{2}}$ to $X \times X$; a natural function here should be the one whose construction can be generalized to any finite set X.

   (b1) Write down all elements (functions) of $\mathbf{2}^X$ explicitly; this means, for each element (function), you must specify its value in the codomain at each point in the domain.
   (b2) Find a natural bijective function from $\mathbf{2}^X$ to the power set $\mathcal{P}(X)$; a natural function here should be the one whose construction can be generalized to any finite set X.

2. Let $a, b, n \in \mathbb{N}$ be natural numbers.

   (a) Use induction to prove the Binomial Formula

   $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

   where $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ (called a *binomial coefficient*) and $0! := 1$ by definition.

   (b) Use the formula to show that

   $$n^{1/n} - 1 < \sqrt{\frac{2}{n}}, \quad n \in \mathbb{N} \text{ and } n \geq 2.$$

   You may begin by writing $n^{1/n} = 1 + \varepsilon$ for some $\varepsilon > 0$, and raise both sides to the power n.