Proceedings of Symposia in Pure Mathematics
Volume **109**, 2024
https://doi.org/10.1090/pspum/109/01993

A note on PL-disks and rationally slice knots

Kristen Hendricks, Jennifer Hom, Matthew Stoffregen, and Ian Zemke

Abstract. We give infinitely many examples of manifold-knot pairs (Y, J) such that Y bounds an integer homology ball, J does not bound a non-locally-flat PL-disk in any integer homology ball, but J does bound a smoothly embedded disk in a rational homology ball. The proof relies on formal properties of involutive Heegaard Floer homology.

Every knot K in S^3 bounds a non-locally-flat PL-embedded disk in B^4, obtained by taking the cone over K. (Throughout, we will not require PL-disks to be locally-flat.) The analogous statement does not hold for knots in more general manifolds. Adam Levine [**9**, Theorem 1.2] found examples of manifold-knot pairs (Y, J) such that Y bounds a contractible 4-manifold and J does not bound a PL-disk in any homology ball X with $\partial X = Y$; see also [**7**].

The main result of this note concerns rationally slice knots in homology spheres bounding integer homology balls:

Theorem 1. *There exist infinitely many manifold-knot pairs (Y, J) where Y is an integer homology sphere and*

(1) *Y bounds an integer homology 4-ball,*
(2) *J does not bound a PL-disk in any integer homology 4-ball,*
(3) *J does bound a smoothly embedded disk in a rational homology 4-ball.*

Throughout, let Y be an integer homology sphere. Recall that a knot $J \subset Y$ is *rationally slice* if J bounds a smoothly embedded disk in a rational homology 4-ball W with $\partial W = Y$. Two manifold-knot pairs (Y_0, J_0) and (Y_1, J_1) are *integrally* (respectively *rationally*) *homology concordant* if J_0 and J_1 are concordant in an integral (respectively rational) homology cobordism between Y_0 and Y_1. A knot $J \subset Y$ is integrally (respectively rationally) homology concordant to a knot K in S^3 if and only if $J \subset Y$ bounds a PL-disk in an integer (respectively rational) homology ball.

Theorem 1 is an immediate consequence of the following theorem, where $\underline{V}_0$ and $\overline{V}_0$ are the involutive knot Floer homology invariants of [**5**] and V_0 the knot Floer homology invariant defined in [**10**, Section 2.2] (see also [**13**], [**11**]):

2020 *Mathematics Subject Classification.* Primary 57K10, 57K18.

The first author was partially supported by NSF grant DMS-2019396 and a Sloan Research Fellowship. The second author was partially supported by NSF grants DMS-1552285 and DMS-2104144. The third author was partially supported by NSF grant DMS-1702532. The fourth author was partially supported by NSF grant DMS-1703685.

Theorem 2. *Let K be a negative amphichiral rationally slice knot in S^3 with $\underline{V}_0 \geq 1$ and $V_0 = \overline{V}_0 = 0$ and let μ be the core of surgery in $M = S^3_{-1/\ell}(K)$, where ℓ is an odd positive integer. Consider $J = \mu \# U \subset M \# -M$, where U denotes the unknot in $-M$. Then $(M \# -M, J)$ is rationally slice, hence rationally homology concordant to a knot in S^3, but $(M \# -M, J)$ is not integrally homology concordant to any knot in S^3.*

Remark 3. The figure-eight satisfies the hypotheses of Theorem 2 by [**3**] (see also [**1**, Section 3] and [**5**, Theorem 1.7]). More generally, the genus one knots K_n with n positive full twists in one band and n negative full twists in the other band, n odd, also satisfy the hypotheses of Theorem 2; see Figure 1. By [**2**, Theorem 4.16], K_n is rationally slice. (Alternatively, K_n is strongly negative amphichiral, hence rationally slice [**8**, Section 2].) Furthermore, $\sigma(K_n) = 0$ since K_n is amphichiral. The knot K_n has Seifert form

$$\begin{pmatrix} n & 1 \\ 0 & -n \end{pmatrix}$$

which implies that $\operatorname{Arf}(K_n) = 1$ if and only if n is odd. Since K_n is alternating, it now follows from [**5**, Theorem 1.7] that for n odd, $\underline{V}_0 = 1$ and $V_0 = \overline{V}_0 = 0$.

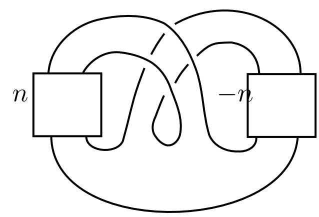

Figure 1. The knot K_n, where n and $-n$ denote the number of positive full twists.

Remark 4. Note that M does not bound an integer homology ball (since, for instance, $\underline{d}(M) = 2\underline{V}_0 \neq 0$), but $M \# -M$ does.

The proof of Theorem 2 is inspired by the proof of [**7**, Theorem 1.1(1)]. Our proof relies on the following result from [**4**] relating the involutive correction term $\underline{d}$ [**5**, Section 5] with the ordinary Heegaard Floer correction term d [**12**, Section 4], for even denominator surgery on knots in S^3:

Proposition 5 ([**4**, Proposition 1.7]). *Let K be a knot in S^3 and let $p, q > 0$ be relatively prime integers, with p odd and q even. Then*

$$\underline{d}(S^3_{p/q}(K), [p/2q]) = d(S^3_{p/q}(K), [p/2q])$$

where $[p/2q]$ denotes the unique self-conjugate Spin^c *structure on $S^3_{p/q}(K)$.*

The key feature from the above proposition is that for even denominator surgery on a knot in S^3, we have that $\underline{d}$ is equal to d for the unique self-conjugate Spin^c structure on the surgery. More generally, we have the following corollary of Proposition 5:

COROLLARY 6. *Let J be a knot in an integer homology sphere Y and let $p, q > 0$ be relatively prime integers, with p odd and q even. If (Y, J) is integrally homology concordant to a knot in S^3, then*

$$\underline{d}(Y_{p/q}(J), [p/2q]) = d(Y_{p/q}(J), [p/2q])$$

where $[p/2q]$ denotes the unique self-conjugate Spin^c *structure on $Y^3_{p/q}(J)$.*

PROOF. If (Y, J) is integrally homology concordant to a knot (S^3, K), then $Y^3_{p/q}(J)$ and $S^3_{p/q}(K)$ are integrally homology cobordant; the homology cobordism is given by surgering along the concordance annulus from (Y, J) to (S^3, K). Since $\underline{d}$ and d are invariants of integer homology cobordism, the result follows from Proposition 5. □

The proof of Theorem 2 relies on finding manifold-knot pairs (Y, J) where $\underline{d}$ and d of even denominator surgery along J differ; the result then follows from Corollary 6.

PROOF OF THEOREM 2. We first show that $(M \# -M, J)$ is rationally slice. Since K is rationally slice, the core of surgery in $M = S^3_{-1/\ell}(K)$ is rationally homology concordant to the core of surgery in $S^3_{-1/\ell}(U)$, which is the unknot in S^3; that is, (M, μ) is rationally slice. Hence $(M \# -M, J)$ is also rationally slice.

We now show that $(M \# -M, J)$ is not integrally homology concordant to any knot in S^3. Since μ is the core of surgery in $S^3_{-1/\ell}(K)$, we have that

$$M_{1/n}(\mu) = S^3_{1/(n-\ell)}(K).$$

Choose an even positive integer n such that $n > \ell$. Since ℓ is odd, n is even, and $n - \ell > 0$, by [**4**, Proposition 1.7] we have that

$$\underline{d}(M_{1/n}(\mu)) = \underline{d}(S^3_{1/(n-\ell)}(K)) = -2\underline{V}_0(K)$$

and

$$d(M_{1/n}(\mu)) = d(S^3_{1/(n-\ell)}(K)) = -2V_0(K) = 0.$$

Since $J = \mu \# U \subset M \# -M$, we have that

$$(M \# -M)_{1/n}(J) = M_{1/n}(\mu) \# -M.$$

Note that $-M = S^3_{1/\ell}(-K) = S^3_{1/\ell}(K)$, where the last equality follows from the fact that K is negative amphichiral. Since $\ell > 0$, [**4**, Proposition 1.7] implies that

$$\underline{d}(-M) = -2\underline{V}_0(K) \quad \text{and} \quad d(-M) = \overline{d}(-M) = 0.$$

Recall that [**6**, Proposition 1.3] states that if Y_1 and Y_2 are integer homology spheres, then

$$\underline{d}(Y_1 \# Y_2) \leq \underline{d}(Y_1) + \overline{d}(Y_2).$$

Hence $\underline{d}(M_{1/n}(\mu) \# -M) \leq -2\underline{V}_0(K)$. Since d is additive under connected sum, we have that $d(M_{1/n}(\mu) \# -M) = 0$.

We have shown that

$$\underline{d}((M \# -M)_{1/2}(J)) \leq -2\underline{V}_0(K) \quad \text{and} \quad d((M \# -M)_{1/2}(J)) = 0.$$

Recall that $\underline{V}_0(K) \geq 1$. Now by Corollary 6, it follows that $(M \# -M, J)$ is not integrally homology concordant to any knot in S^3. □

Acknowledgments

We thank JungHwan Park for helpful conversations and Chuck Livingston for thoughtful comments on an earlier draft.

References

[1] Selman Akbulut and Kyle Larson, *Brieskorn spheres bounding rational balls*, Proc. Amer. Math. Soc. **146** (2018), no. 4, 1817–1824, DOI 10.1090/proc/13828. MR3754363

[2] Jae Choon Cha, *The structure of the rational concordance group of knots*, Mem. Amer. Math. Soc. **189** (2007), no. 885, x+95, DOI 10.1090/memo/0885. MR2343079

[3] Ronald Fintushel and Ronald J. Stern, *A μ-invariant one homology* 3*-sphere that bounds an orientable rational ball*, Four-manifold theory (Durham, N.H., 1982), Contemp. Math., vol. 35, Amer. Math. Soc., Providence, RI, 1984, pp. 265–268, DOI 10.1090/conm/035/780582. MR780582

[4] Kristen Hendricks, Jennifer Hom, Matthew Stoffregen, and Ian Zemke, *Surgery exact triangles in involutive Heegaard Floer homology*, 2020, Preprint, `arXiv:2011.00113`.

[5] Kristen Hendricks and Ciprian Manolescu, *Involutive Heegaard Floer homology*, Duke Math. J. **166** (2017), no. 7, 1211–1299, DOI 10.1215/00127094-3793141. MR3649355

[6] Kristen Hendricks, Ciprian Manolescu, and Ian Zemke, *A connected sum formula for involutive Heegaard Floer homology*, Selecta Math. (N.S.) **24** (2018), no. 2, 1183–1245, DOI 10.1007/s00029-017-0332-8. MR3782421

[7] Jennifer Hom, Adam Simon Levine, and Tye Lidman, *Knot concordance in homology cobordisms*, Duke Math. J. **171** (2022), no. 15, 3089–3131, DOI 10.1215/00127094-2021-0110. MR4497224

[8] Akio Kawauchi, *Rational-slice knots via strongly negative-amphicheiral knots*, Commun. Math. Res. **25** (2009), no. 2, 177–192. MR2554510

[9] Adam Simon Levine, *Nonsurjective satellite operators and piecewise-linear concordance*, Forum Math. Sigma **4** (2016), Paper No. e34, 47, DOI 10.1017/fms.2016.31. MR3589337

[10] Yi Ni and Zhongtao Wu, *Cosmetic surgeries on knots in S^3*, J. Reine Angew. Math. **706** (2015), 1–17, DOI 10.1515/crelle-2013-0067. MR3393360

[11] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), no. 1, 58–116, DOI 10.1016/j.aim.2003.05.001. MR2065507

[12] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261, DOI 10.1016/S0001-8708(02)00030-0. MR1957829

[13] Jacob Andrew Rasmussen, *Floer homology and knot complements*, ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–Harvard University. MR2704683

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08854
Email address: `kristen.hendricks@rutgers.edu`

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332
Email address: `hom@math.gatech.edu`

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824
Email address: `stoffre1@msu.edu`

Department of Mathematics, Princeton University, Princeton, New Jersey 08544
Email address: `izemke@math.princeton.edu`